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A UNIFIED VIEW OF SOME VERTEX OPERATOR CONSTRUCTIONS

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ABSTRACT

We present a general vertex operator construction based on the Fock space for affine Lie algebras of type A. This construction allows us to give a unified treatment for both the homogeneous and principle realizations of the affine Lie algebras $\hat{gl}_N$ as well as for some extended affine Lie algebras coordinatized by certain quantum tori.

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0. Introduction

This paper presents a unified view of certain vertex operator constructions for some of the extended affine Lie algebras (EALA's for short) which are coordinatized by certain quantum tori. Recall that for the affine Kac–Moody Lie algebras vertex operator representations were developed in [LW] and [KKLW] for the principal realizations and in [FK], [S] in the homogeneous realization. Our motivation comes from the paper [F1] of I. Frenkel, where he presented a unified construction for both the principal and homogeneous realizations of the affine Lie algebras of type $A^{(1)}$. This is accomplished by using the affine algebra $\hat{gl}_M$ rather than $\hat{sl}_M$. Moreover, Frenkel used a Clifford algebra structure, which was inherent in his situation, to define a new type of normal ordering which then led to his unified view. The Clifford structure had been studied before in the works [F3,4] and [KP].

The structure theory of EALA's has been developed over the last ten years (see [H-KT], [BGK],[AABGP] and [ABGP]). Roughly speaking these Lie algebras are generalizations of both the affine Kac–Moody Lie algebras and the finite-dimensional simple Lie algebras over the complex numbers which admit Laurent-like coordinates in a finite number of variables. It turns out that algebras of different types admit different types of coordinates. For example, those of type A_l admit the non-commutative quantum torus as coordinates (see [M], [BGK]). Representations for these Lie algebras over quantum tori have been constructed in [JK], [G-KK], [G1,2,3], [BS], [VV]. When $l = 1$ there are even Jordan algebras which serve as coordinates of EALAs. The study of representations for this type of Lie algebra has been initiated in [T1]. Perhaps the examples which have attracted the most attention so far are the toroidal algebras which have the commutative associative Laurent polynomials as their coordinates. The toroidal algebras have been studied since the mid-80's both in terms of their structure theory as well as their representation theory (see for example [F2], [MRY], [Y], [W], [BC], [FJW], [T2]). For our purposes we want to mention that vertex operator representations have played a predominant role in much of this work. Indeed, in [G1,2], one finds both homogeneous and principal realizations given for many of the EALA's of type A. The principal realization for those EALAs was also implicitly given by [G-KK] in studying the so-called Γ-conformal algebras. Our goal in this work is to unify the various approaches and show how they all follow from the same type of approach. Of course, the work in the affine case, namely [F1], shed light on doing this.

Working with a standard type of Fock space we are able to define a general type of vertex operator which depends on a non-zero scalar from $\mathbb{C}$ and to then compute the commutator of two of these operators. This is presented in the second section of this paper while, in the first section, we give the basics on the Lie algebras, which are all of type A, which we will later go on to find representations for. Already in Section 1 it is evident that there is somewhat of a unified picture for these algebras. When we define our vertex operators in Section 2 the reader will see that we are using a Clifford algebra structure to define the normal ordering we are using, just as was done in [F1]. In the third section we introduce some Lie algebras associated to certain choices of subgroups, G, of non-zero complex numbers as well as the choice of a positive integer M. These Lie algebras are spanned by the moments of our vertex operators and hence, by construction, we automatically have a representation for this Lie algebra. We show the representations we have are completely reducible and find the irreducible components. In the fourth and final section we show how certain choices of the group G and the integer M lead to representations of the algebras of Section 1. Thus, we recover both the principal and homogeneous representations for the affine algebras of type A as well as those for the EALA's studied in [BS], [G1,2], [G-KK]. It is from seeing the various applications in this fourth section that one understands the unification of our treatment. Finally, we want to emphasize that this unified treatment would not have been possible without first knowing the particular special cases of this result.

1. Preliminaries

In this section we shall review some of the basics on Lie algebras coordinatized by quantum tori. We present this from a general point of view which unifies our treatment. For notation we always denote the integers, positive and negative integers respectively by $\mathbb{Z}$, $\mathbb{Z}_+$ and $\mathbb{Z}_-$.

Let $\mathfrak{g}$ be any associative $\mathbb{C}$-algebra with a symmetric bilinear form $(\cdot,\cdot)$: $\mathfrak{g}\times\mathfrak{g}\to\mathbb{C}$ such that $(xy,z)=(x,yz)$ for $x,y,z\in\mathfrak{g}$. Let $A=\bigoplus_{\alpha\in\mathbb{Z}^{\nu+1}}A_\alpha$, $\nu\geq 0$, be any $\mathbb{Z}^{\nu+1}$-graded associative algebra such that $\dim A_\alpha<\infty$ for all $\alpha\in\mathbb{Z}^{\nu+1}$. Fix a base $(x_{i\alpha})_{i\in I_\alpha}$ of A_α, where I_α is the index set corresponding to the subspace A_α. Let $d_0,d_1,\ldots,d_\nu$ be degree derivations of A such that $d_ix=\alpha_ix$ for $x\in A_\alpha$, $i=0,1,\ldots,\nu$ and $\alpha=(\alpha_0,\ldots,\alpha_\nu)\in\mathbb{Z}^{\nu+1}$. We define a $\mathbb{C}$-linear map ϕ: $A\to\mathbb{C}$ by linear extension of

$$\phi(x_{i\alpha})=\begin{cases}1 & \text{if } \alpha=(0,\ldots,0)\\ 0 & \text{otherwise}\end{cases}\tag{1.1}$$

for $i\in I_\alpha$, $\alpha\in\mathbb{Z}^{\nu+1}$. The tensor product $\mathfrak{g}\otimes_{\mathbb{C}}A$, with the canonical product

$(x \otimes a)(y \otimes b) = xy \otimes ab$ for $x, y \in \mathfrak{g}, a, b \in A$, is also an associative algebra. Moreover, with the commutator product

$$[x \otimes a, y \otimes b]_{loop} = (x \otimes a)(y \otimes b) - (y \otimes b)(x \otimes a),$$

$\mathfrak{g} \otimes_{\mathbb{C}} A$ forms a $\mathbb{Z}^{\nu+1}$-graded Lie algebra. We call this algebra a loop-type Lie algebra. Consider the vector space

$$\hat{\mathfrak{g}}_A := (\mathfrak{g} \otimes_{\mathbb{C}} A) \oplus \mathcal{C} \tag{1.2}$$

where $\mathcal{C}=\bigoplus_{0\le i\le\nu} \mathbb{C}c_i$ is a $(\nu+1)$-dimensional vector space. There is an alternating bilinear map $[\cdot,\cdot]$: $\hat{\mathfrak{g}}_A \times \hat{\mathfrak{g}}_A \to \hat{\mathfrak{g}}_A$ determined by the conditions

$$[c_i, \hat{\mathfrak{g}}_A] = 0,$$
$$[x \otimes a, y \otimes b] = [x \otimes a, y \otimes b]_{loop} + (x, y) \sum_{0\le i\le\nu} \phi((d_i a)b)c_i,$$

for $x, y \in \mathfrak{g}, a, b \in A$ and $i = 0, 1, \ldots, \nu$. It is straightforward to check that $\hat{\mathfrak{g}}_A$ is a Lie algebra. Indeed, there is an exact sequence of Lie algebras with canonical maps

$$0 \to \bigoplus_{0\le i\le\nu} \mathbb{C}c_i \to \hat{\mathfrak{g}}_A \to \mathfrak{g} \otimes_{\mathbb{C}} A \to 0,$$

and so we have that $\hat{\mathfrak{g}}_A$ is a central extension of the loop-type Lie algebra $\mathfrak{g} \otimes_{\mathbb{C}} A$.

Let $M_\infty(\mathbb{C}) = \text{span}_{\mathbb{C}}\{E_{ij} | 1 \le i, j < \infty\}$ be the infinite matrix algebra, where E_{ij} is the infinite matrix with a 1 in the (i, j)-entry and zero's elsewhere. We also let $M_n(\mathbb{C}) = \text{span}_{\mathbb{C}}\{E_{ij} | 1 \le i, j \le n\}$. This subspace of $M_\infty(\mathbb{C})$ for $n \ge 1$ is isomorphic to the usual matrix algebra of $n \times n$ matrices with entries in $\mathbb{C}$.

Let $Q = (q_{ij})$ be a $(\nu+1) \times (\nu+1)$ matrix with entries $q_{ij} \in \mathbb{C}^\times$ satisfying $q_{ii} = 1$ and $q_{ij} = q_{ji}^{-1}$ for $0 \le i, j \le \nu$. The quantum torus associated with the matrix Q is a unital associative $\mathbb{C}$-algebra $\mathbb{C}_Q := \mathbb{C}_Q[t_0^{\pm1}, \ldots, t_\nu^{\pm1}]$ with generators $t_0^{\pm1}, \ldots, t_\nu^{\pm1}$ and relations $t_i t_i^{-1} = t_i^{-1} t_i = 1, \quad t_i t_j = q_{ij} t_j t_i$, for $0 \le i, j \le \nu$. If Q is a 2×2 matrix, so then $\nu = 1$, the matrix $Q = (q_{ij})$ is determined by a single $q = q_{10}$. In this case we often simply denote $\mathbb{C}_Q = \mathbb{C}_Q[t_0^{\pm1}, t_1^{\pm1}]$ by $\mathbb{C}_q$. Choose the bilinear form on $M_n(\mathbb{C})$ to be the trace form. Set $A = \mathbb{C}_Q[t_0^{\pm1}, \ldots, t_\nu^{\pm1}]$, with the $\mathbb{Z}^{\nu+1}$-gradation $A = \bigoplus_{\alpha\in\mathbb{Z}^{\nu+1}} A_\alpha$, where the subspace A_α is spanned by $t^\alpha = t_0^{\alpha_0} t_1^{\alpha_1} \cdots t_\nu^{\alpha_\nu}$ for $\alpha = (\alpha_0, \ldots, \alpha_\nu) \in \mathbb{Z}^{\nu+1}$. Define σ_Q: $\mathbb{Z}^{\nu+1} \times \mathbb{Z}^{\nu+1} \to \mathbb{C}$ by

$$\sigma_Q(\alpha, \beta) = \prod_{0\le i<j\le\nu} q_{ji}^{\alpha_j \beta_i} \tag{1.3}$$

for $\alpha = (\alpha_0, \ldots, \alpha_\nu)$, $\beta = (\beta_0, \ldots, \beta_\nu) \in \mathbb{Z}^{\nu+1}$. Then we have

$$t^\alpha t^\beta = \sigma_Q(\alpha, \beta) t^{\alpha+\beta}.$$

The proof of the following Lemma is clear.

LEMMA 1.4: *Let $m, n \geq 1$ be integers. Then there is a Lie algebra isomorphism*

$$(M_m(\mathbb{C}) \otimes M_n(\mathbb{C}))^\wedge_{\mathbb{C}_Q} \cong (M_{mn}(\mathbb{C}))^\wedge_{\mathbb{C}_Q}$$

which is given by

$$E_{ij} \otimes E_{kl} \otimes t^\alpha \mapsto E_{(i-1)n+k,(j-1)n+l} \otimes t^\alpha,$$
$$c_s \mapsto c_s, \quad s = 0, 1, \ldots, \nu,$$

for $\alpha = (\alpha_0, \ldots, \alpha_\nu) \in \mathbb{Z}^{\nu+1}$, $1 \leq i, j \leq m, 1 \leq k, l \leq n$.

Let $\hat{\mathcal{L}}_{\mathbb{C}_Q}$ be the Lie subalgebra of $(M_m(\mathbb{C}) \otimes M_n(\mathbb{C}))^\wedge_{\mathbb{C}_Q}$ generated by elements of the form $E_{ij} \otimes E_{kl} \otimes t_0^{\alpha_0(n-1)+l-k} t^\alpha$ for $1 \leq i, j \leq m$, $1 \leq k, l \leq n$ and $\alpha = (\alpha_0, \ldots, \alpha_\nu) \in \mathbb{Z}^{\nu+1}$. The following result gives the structure of $\hat{\mathcal{L}}_{\mathbb{C}_Q}$.

PROPOSITION 1.5:

$$\hat{\mathcal{L}}_{\mathbb{C}_Q} \cong (M_m(\mathbb{C}) \otimes M_n(\mathbb{C}))^\wedge_{\mathbb{C}_{Q^*}}$$

where $\mathbb{C}_Q = \mathbb{C}_Q[t_0^{\pm 1}, \ldots, t_\nu^{\pm 1}]$ with $Q = (q_{ij})$, and $\mathbb{C}_{Q^*} = \mathbb{C}_{Q^*}[\tau_0^{\pm 1}, \ldots, \tau_\nu^{\pm 1}]$ with $Q^* = (q^*_{ij})$ such that $q^*_{ij} = q_{ij}$ if $i, j \neq 0$, and $q^*_{ij} = q^n_{ij}$ if $i = 0$ or $j = 0$.

Proof: Define a linear map $f\colon (M_m(\mathbb{C}) \otimes M_n(\mathbb{C}))^\wedge_{\mathbb{C}_{Q^*}} \to \hat{\mathcal{L}}_{\mathbb{C}_Q}$ by

$$E_{ij} \otimes E_{kl} \otimes \tau^\alpha \mapsto \Big(\prod_{1 \leq s \leq \nu} q_{s0}^{l\alpha_s}\Big) E_{ij} \otimes E_{kl} \otimes t_0^{(n-1)\alpha_0+l-k} t^\alpha - k\delta_{ij}\delta_{kl}\delta_{\alpha,0}c_0,$$
$$c_0 \mapsto nc_0, \quad c_s \mapsto c_s, \quad s = 1, 2, \ldots, \nu,$$

for $\alpha = (\alpha_0, \ldots, \alpha_\nu) \in \mathbb{Z}^{\nu+1}$, $1 \leq i, j \leq m$ and $1 \leq k, l \leq n$. Let $\bar{\alpha} = (n\alpha_0 + l - k, \alpha_1, \ldots, \alpha_\nu)$, and $\bar{\alpha}' = (n\alpha'_0 + l' - k', \alpha'_1, \ldots, \alpha'_\nu) \in \mathbb{Z}^{\nu+1}$. Using the identity

$$\sigma_Q(\bar{\alpha}, \bar{\alpha}') = \sigma_{Q^*}(\alpha, \alpha') \prod_{1 \leq j \leq \nu} q_{j0}^{\alpha_j(l'-k')} \tag{1.6}$$

one can easily check that the map f is the desired Lie algebra isomorphism. ∎

Putting together the two previous results we get the following identification of $\hat{\mathcal{L}}_{\mathbb{C}_Q}$.

COROLLARY 1.7:

$$\hat{\mathcal{L}}_{\mathbb{C}_Q} \cong (M_{mn}(\mathbb{C}))^{\wedge}_{\mathbb{C}_{Q^*}}$$

where Q and Q^* are given in the previous proposition .

Let $\xi = \xi_n$ be an n-th primitive root of unity and let $E, F \in M_n(\mathbb{C})$ be defined by saying

$$E = E_{12} + E_{23} + \cdots + E_{n-1,n} + E_{n1}, \quad F = \sum_{i=1}^{n} E_{ii}(\xi^{i-1}). \tag{1.8}$$

Then the following fact is well-known.

LEMMA 1.9: *The set of matrices $\{F^iE^j\}_{1\le i,j\le n}$ forms a basis of the matrix algebra $M_n(\mathbb{C})$ (so a basis of the general linear Lie algebra $gl_n(\mathbb{C})$). Moreover,*

$$EF = \xi FE, \quad E^n = F^n = Id_n, \tag{1.10}$$

and

$$E_{ij} = \frac{1}{n}\sum_{k=0}^{n-1} \xi^{k(1-i)} F^k E^{j-i}, \quad F^iE^j = \sum_{l=1}^{n} \xi^{i(l-1)} E_{l,\overline{l+j}} \tag{1.11}$$

for $1 \le i,j \le n$, where, for notation, we are letting $\bar{l}$ denote the unique integer, l, in $\{1,2,\dots,n\}$ such that $\bar{l} = l(\bmod n)$.

Note that

$$\begin{aligned} E_{ij} \otimes E_{kl} \otimes t_0^{\alpha_0(n-1)+l-k} t^{\alpha} &= \sum_{s=0}^{n-1} \xi^{s(1-k)} E_{ij} \otimes F^s E^{l-k} \otimes t_0^{\alpha_0 n+l-k} t_1^{\alpha_1} \cdots t_\nu^{\alpha_\nu} \\ &= \sum_{s=0}^{n-1} \xi^{s(1-k)} E_{ij} \otimes F^s E^{\alpha_0'} \otimes t_0^{\alpha_0'} t_1^{\alpha_1} \cdots t_\nu^{\alpha_\nu} \end{aligned}$$

where $\alpha_0' = \alpha_0 n + l - k$, for $1 \le i,j \le m$, $1 \le k,l \le n$, and $\alpha = (\alpha_0, \alpha_1, \dots, \alpha_\nu) \in \mathbb{Z}^{\nu+1}$. From this one sees that the Lie subalgebra $\hat{\mathcal{L}}_{\mathbb{C}_Q}$ of $(M_m(\mathbb{C}) \otimes M_n(\mathbb{C}))^{\wedge}_{\mathbb{C}_Q}$ has a basis of the form

$$E_{ij} \otimes F^k E^{l_0} \otimes t_0^{l_0} \cdots t_\nu^{l_\nu}, \quad c_0, c_1, \dots, c_\nu \tag{1.12}$$

where $1 \le i,j \le m$, $0 \le k \le n-1$ and $l_0, l_1, \dots, l_\nu \in \mathbb{Z}$. Moreover, the commutation relations of $\hat{\mathcal{L}}_{\mathbb{C}_Q}$ are determined by

$$[E_{ij} \otimes F^k E^{\alpha_0} \otimes t^{\alpha}, E_{i'j'} \otimes F^{k'} E^{\alpha_0'} \otimes t^{\alpha'}] \tag{1.13}$$

$$
\begin{aligned}
&=\delta_{ji'}\xi^{\alpha_0 k'}\sigma_Q(\alpha,\alpha')E_{ij'}\otimes F^{k+k'}E^{\alpha_0+\alpha'_0}\otimes t^{\alpha+\alpha'}\\
&\quad-\delta_{j'i}\xi^{\alpha'_0 k}\sigma_Q(\alpha',\alpha)E_{i'j}\otimes F^{k+k'}E^{\alpha_0+\alpha'_0}\otimes t^{\alpha+\alpha'}\\
&\quad+n\delta_{ji'}\delta_{ij'}\delta_{\overline{k+k'},0}\delta_{\alpha+\alpha',0}\xi^{\alpha_0 k'}\sum_{0\le s\le\nu}\alpha_s c_s
\end{aligned}
$$

for $1\le i,i',j,j'\le m$, $0\le k,k'\le n-1$, $\alpha=(\alpha_0,\dots,\alpha_\nu)$, $\alpha'=(\alpha'_0,\dots,\alpha'_\nu)\in\mathbb{Z}^{\nu+1}$, as well as the fact that the elements $c_0,\dots,c_\nu$ are central in $\hat{\mathcal{L}}_{\mathbb{C}_Q}$.

From now on we will identify the Lie algebra $(M_m(\mathbb{C})\otimes M_n(\mathbb{C}))^{\wedge}_{\mathbb{C}_Q}$ with $(M_{mn}(\mathbb{C}))^{\wedge}_{\mathbb{C}_Q}$, and also identify $\hat{\mathcal{L}}_{\mathbb{C}_Q}$ with $(M_{mn}(\mathbb{C}))^{\wedge}_{\mathbb{C}_{Q^*}}$, where $Q=(q_{ij})$, $Q^*=(q^*_{ij})$ and, as above, $q^*_{ij}=q_{ij}$ if $i,j\neq 0$, and $q^*_{ij}=q^n_{ij}$ if $i=0$ or $j=0$. For simplicity we will write $a\alpha=(a\alpha_1,\dots,a\alpha_\nu)$ for $a\in\mathbb{Z}$ and $\alpha\in\mathbb{Z}^\nu$; also we will write

$$q_0^\alpha=q_{10}^{\alpha_1}\cdots q_{\nu 0}^{\alpha_\nu}$$

for $q_0=(q_{10},\dots,q_{\nu 0})\in\mathbb{C}^\nu$.

The Lie algebra structure (1.13) of $\hat{\mathcal{L}}_{\mathbb{C}_Q}$ can be described by formal power series identities. For this purpose we let z,z_1,z_2 be formal variables. For $1\le i,j\le m$, $0\le k\le n-1$ and $\alpha=(\alpha_1,\dots,\alpha_\nu)\in\mathbb{Z}^\nu$, we set

$$X_{ij}^k(\alpha,z)=\sum_{l\in\mathbb{Z}}(E_{ij}\otimes F^kE^l\otimes t_0^l t_1^{\alpha_1}\cdots t_\nu^{\alpha_\nu})z^{-l}\in\hat{\mathcal{L}}_{\mathbb{C}_Q}[[z,z^{-1}]], \tag{1.14}$$

and $\delta(z)=\sum_{l\in\mathbb{Z}}z^l$, $(D\delta)(z)=\sum_{l\in\mathbb{Z}}lz^l$. Then the algebra structure of $\hat{\mathcal{L}}_{\mathbb{C}_Q}$ is described by the following lemma.

LEMMA 1.15: *Let $1\le i,j,i',j'\le m$, $0\le k,k'\le n-1$, $\alpha=(\alpha_1,\dots,\alpha_\nu)$, $\alpha'=(\alpha'_1,\dots,\alpha'_\nu)\in\mathbb{Z}^\nu$. Then the following power series identity is equivalent to (1.13):*

$$
\begin{aligned}
[X_{ij}^k(\alpha,z_1),X_{i'j'}^{k'}(\alpha',z_2)]&=\delta_{ji'}\sigma(\alpha,\alpha')X_{ij'}^{\overline{k+k'}}(\alpha+\alpha',\xi^{-k'}z_1)\delta\Big(\frac{\xi^{k'}z_2}{z_1q_0^\alpha}\Big)\\
&\quad-\delta_{j'i}\sigma(\alpha',\alpha)X_{i'j}^{\overline{k+k'}}(\alpha+\alpha',\xi^{-k}z_1)\delta\Big(\frac{z_2q_0^{\alpha'}}{\xi^k z_1}\Big)\\
&\quad+n\delta_{ji'}\delta_{ij'}\delta_{\overline{k+k'},0}\delta_{\alpha+\alpha',0}\sigma(\alpha,\alpha')\Big\{(D\delta)\Big(\frac{\xi^{k'}z_2}{z_1q_0^\alpha}\Big)c_0+\delta\Big(\frac{\xi^{k'}z_2}{z_1q_0^\alpha}\Big)\sum_{1\le s\le\nu}\alpha_s c_s\Big\}
\end{aligned}
\tag{1.16}
$$

where $\bar{k}=k(\mathrm{mod}\,n)$ and $k\in\{0,1,\dots,n-1\}$.

As very special cases, one chooses $n=1$, $\nu=0$, in which case $\hat{\mathcal{L}}_{\mathbb{C}_Q}$ is just the affine algebra $\widehat{gl}_m(\mathbb{C})$ in the so-called homogeneous picture; while if one chooses

$m = 1$, $\nu = 0$ then $\hat{\mathcal{L}}_{\mathbb{C}_Q}$ is the affine algebra $\widehat{gl}_n(\mathbb{C})$ in the so-called principal picture. In these two cases, the identity (1.16) can simply be written as follows:
(1.17)

$$[X^0_{ij}(z_1), X^0_{kl}(z_2)] = X^0_{il}(z_1)\delta_{jk}\delta\Big(\frac{z_2}{z_1}\Big) - X^0_{kj}(z_2)\delta_{il}\delta\Big(\frac{z_2}{z_1}\Big) + \delta_{jk}\delta_{il}(D\delta)\Big(\frac{z_2}{z_1}\Big)c_0,$$

for $1 \le i, j, k, l \le m$, for the first of these and

$$\begin{aligned}[X^i_{11}(z_1), X^j_{11}(z_2)] =& X^{\overline{i+j}}_{11}(z_2)\delta\Big(\frac{\xi^j z_2}{z_1}\Big) - X^{\overline{i+j}}_{11}(z_1)\delta\Big(\frac{\xi^i z_1}{z_2}\Big)\\ &+ n\delta_{\overline{i+j},0}(D\delta)\Big(\frac{\xi^j z_2}{z_1}\Big)c_0.\end{aligned} \tag{1.18}$$

for $0 \le i, j \le n-1$, $\bar{i} = i(\mathrm{mod}\, n)$ and $i \in \{0, 1, \ldots, n-1\}$ for the second one.

Moreover, if we choose $n = 1, \nu = 1$, or $m = 1, \nu = 1$, and write $q_{10} = q$, then $\hat{\mathcal{L}}_{\mathbb{C}_Q}$ gives respectively the homogeneous realization of the Lie algebra $\widehat{gl}_m(\mathbb{C}_q)$, and the principal realization of $\widehat{gl}_n(\mathbb{C}_{q^n})$. The algebra structure of these two cases can be described as follows:

$$\begin{aligned}[X^0_{ij}(r, z_1), X^0_{kl}(s, z_2)] =& X^0_{il}(r+s, z_1)\delta_{jk}\delta\Big(\frac{z_2}{q^r z_1}\Big) - X^0_{kj}(r+s, z_2)\delta_{il}\delta\Big(\frac{q^s z_2}{z_1}\Big)\\ &+ \delta_{il}\delta_{jk}\delta_{r+s,0}\Big((D\delta)\Big(\frac{z_2}{q^r z_1}\Big)c_0 + r\delta\Big(\frac{z_2}{q^r z_1}\Big)c_1\Big)\end{aligned} \tag{1.19}$$

for $1 \le i, j, k, l \le m$ and $r, s \in \mathbb{Z}$, for the first and

$$\begin{aligned}[X^i_{11}(r, z_1), X^j_{11}(s, z_2)] =& X^{\overline{i+j}}_{11}(r+s, \xi^{-j}z_1)\delta\Big(\frac{\xi^j z_2}{q^r z_1}\Big)\\ &- X^{\overline{i+j}}_{11}(r+s, \xi^{-i}z_2)\delta\Big(\frac{q^s z_2}{\xi^i z_1}\Big)\\ &+ n\delta_{r+s,0}\delta_{\overline{i+j},0}\Big((D\delta)\Big(\frac{\xi^j z_2}{q^r z_1}\Big)c_0 + r\delta\Big(\frac{\xi^j z_2}{q^r z_1}\Big)c_1\Big)\end{aligned} \tag{1.20}$$

for $1 \le i, j \le n$, $r, s \in \mathbb{Z}$, and $\overline{i+j} = i+j \ (\mathrm{mod}\, n)$ for the second.

Finally, if we choose $\nu = 1$, $m, n \ge 1$, write $q_{10} = q$, in which case $\hat{\mathcal{L}}_{\mathbb{C}_Q}$ is isomorphic to the affine Lie algebra $\widehat{gl}_{mn}(\mathbb{C}_{q^n})$, which contains the special cases mentioned above. The algebra structure is as follows:

$$\begin{aligned}[X^k_{ij}(r, z_1), X^{k'}_{i'j'}(r', z_2)] =& \delta_{ji'}X^{\overline{k+k'}}_{ij'}(r+r', \xi^{-k'}z_1)\delta\Big(\frac{\xi^{k'} z_2}{q^r z_1}\Big)\\ &- \delta_{j'i}X^{\overline{k+k'}}_{i'j}(r+r', \xi^{-k}z_2)\delta\Big(\frac{q^{r'} z_2}{\xi^k z_1}\Big)\\ &+ n\delta_{ji'}\delta_{ij'}\delta_{\overline{k+k'},0}\delta_{r+r',0}\Big\{(D\delta)\Big(\frac{\xi^{k'} z_2}{q^r z_1}\Big)c_0 + r\delta\Big(\frac{\xi^{k'} z_2}{q^r z_1}\Big)c_1\Big\}.\end{aligned} \tag{1.21}$$

In subsequent sections we are going to give irreducible representations for a class of Lie algebras which include the Lie algebras mentioned above.

2. Fock space and vertex operators

In this section, we shall define the Fock space we need and construct a family of vertex operators acting on it. Then we go on to derive the commutation relations between these vertex operators in various situations. Some of these commutation relations were implicitly worked out in [G1]. However, we will use the ideas from [F1] to tie a Clifford algebra structure to our vertex operators. This makes our approach very natural and concise.

Let $\varepsilon_1, \ldots, \varepsilon_M$ $(M \geq 1)$ be symbols. We form lattices

$$\Gamma_M = \bigoplus_{i=1}^{M} \mathbb{Z}\varepsilon_i, \quad Q_M = \bigoplus_{i=1}^{M-1} \mathbb{Z}(\varepsilon_i - \varepsilon_{i+1}), \tag{2.1}$$

with a symmetric bilinear form $(\varepsilon_i, \varepsilon_j) = \delta_{ij}$. We also extend this bilinear form to the $\mathbb{C}$-vector space

$$H_M := \mathbb{C} \otimes \Gamma_M. \tag{2.2}$$

For each $k \in \mathbb{Z}$ we take a copy of H_M with basis labelled by $\varepsilon_i(k)$ for $1 \leq i \leq M$, $k \in \mathbb{Z}$. That is, $\varepsilon_i(k)$ is to be a copy of ε_i. We form a Lie algebra

$$\mathcal{H}_M = \mathrm{span}_{\mathbb{C}}\{\varepsilon_i(k), c | 1 \leq i \leq M, k \in \mathbb{Z}\}, \tag{2.3}$$

with the Lie product

$$[\alpha(k), \beta(l)] = k(\alpha, \beta)\delta_{k+l,0}c, \tag{2.4}$$

for $\alpha, \beta \in H_M$, $k, l \in \mathbb{Z}$, where c is a central element. Let

$$\mathcal{H}_M^{\pm} = \mathrm{span}\{\varepsilon_i(k) | k \in \mathbb{Z}_{\pm}, 1 \leq i \leq M\}. \tag{2.5}$$

Then

$$\hat{\mathcal{H}}_M = \mathcal{H}_M^{+} + \mathbb{C}c + \mathcal{H}_M^{-} \tag{2.6}$$

forms a Heisenberg subalgebra of $\mathcal{H}_M$.

Let $\mathcal{S}(\mathcal{H}_M^{-})$ be the symmetric algebra over the abelian algebra $\mathcal{H}_M^{-}$ and let

$$\mathbb{C}[\Gamma_M] := \bigoplus_{\alpha \in \Gamma_M} \mathbb{C}e^{\alpha}, \tag{2.7}$$

be the group algebra over Γ_M twisted by a 2-cocycle so that $e^\alpha e^\beta = \epsilon(\alpha,\beta)e^{\alpha+\beta}$ for $\alpha,\beta\in\Gamma_M$. The cocycle

$$\epsilon\colon \Gamma_M\times\Gamma_M\to\{\pm 1\}, \tag{2.8}$$

is defined by setting

$$\epsilon(\varepsilon_i,\varepsilon_j)=1 \text{ if } i\le j,\quad \epsilon(\varepsilon_i,\varepsilon_j)=-1 \text{ if } i>j, \tag{2.9}$$

and

$$\epsilon\Big(\sum_i m_i\varepsilon_i,\sum_j n_j\varepsilon_j\Big)=\prod_{i,j}(\epsilon(\varepsilon_i,\varepsilon_j))^{m_in_j}, \tag{2.10}$$

for $m_i,n_j\in\mathbb{Z}$. One can easily check the following result.

LEMMA 2.11: *ϵ is bi-multiplicative on Γ_M. Moreover,*

$$\epsilon(\alpha,\alpha)=(-1)^{(\alpha,\alpha)/2},\quad \epsilon(\alpha,\beta)\epsilon(\beta,\alpha)=(-1)^{(\alpha,\beta)}$$

for $\alpha,\beta\in Q_M$.

Now we define the Fock space

$$V_M=\mathcal{S}(\mathcal{H}_M^-)\otimes\mathbb{C}[\Gamma_M] \tag{2.12}$$

which affords representations for both the Lie algebra $\mathcal{H}_M$ and the group algebra $\mathbb{C}[\Gamma_M]$ with the following actions:

$$\begin{aligned}
&\varepsilon_i(k).u\otimes e^\beta = k\Big(\frac{\partial}{\partial\varepsilon_i(-k)}u\Big)\otimes e^\beta,\quad \text{for } k\in\mathbb{Z}_+,\\
&\varepsilon_i(k).u\otimes e^\beta = (\varepsilon_i(k)u)\otimes e^\beta,\quad \text{for } k\in\mathbb{Z}_-,\\
&\varepsilon_i(0).u\otimes e^\beta = (\varepsilon_i,\beta)u\otimes e^\beta,\\
&c.u\otimes e^\beta = u\otimes e^\beta,\quad\text{and}\quad e^\alpha.u\otimes e^\beta=\epsilon(\alpha,\beta)u\otimes e^{\alpha+\beta},
\end{aligned}$$

for $\alpha,\beta\in\Gamma_M$, $1\le i\le M$, and $u\in\mathcal{S}(\mathcal{H}_M^-)$. For $\alpha\in\Gamma_M$, we define (we are using the standard notation from [FLM])

$$\alpha(z)=\sum_{k\in\mathbb{Z}}\alpha(k)z^{-k}\in(\operatorname{End}V_M)[[z,z^{-1}]] \tag{2.13}$$

and

$$E^\pm(\alpha,z)=\exp\Big(\sum_{k\in\mathbb{Z}_\pm}\frac{\alpha(k)}{k}z^{-k}\Big)\in(\operatorname{End}V_M)[[z,z^{-1}]]. \tag{2.14}$$

Then the following lemma is straightforward.

LEMMA 2.15: *For* $\alpha, \beta \in \Gamma_M$, $a, b \in \mathbb{C}^\times := \mathbb{C} \setminus \{0\}$, *we have*

$$E^{\pm}(0, az) = 1, [\alpha(k), E^{+}(\beta, az)] = 0, \quad \text{if } k \geq 0,$$

$$[\alpha(z_1), E^{\pm}(\beta, az_2)] = -(\alpha, \beta) E^{\pm}(\beta, az_2) \sum_{k \in \mathbb{Z}_{\mp}} \Big(\frac{az_2}{z_1}\Big)^k,$$

$$E^{\pm}(\alpha, az) E^{\pm}(\beta, bz) = \exp\Big(\sum_{k \in \mathbb{Z}_{\pm}} \frac{1}{k} (a^{-k}\alpha(k) + b^{-k}\beta(k)) z^{-k} \Big),$$

$$E^{+}(\alpha, az_1) E^{-}(\beta, bz_2) = E^{-}(\beta, bz_2) E^{+}(\alpha, az_1) \Big(1 - \frac{bz_2}{az_1}\Big)^{(\alpha,\beta)}.$$

Let $v = \alpha_1(-1)^{k_1} \cdots \alpha_r(-r)^{k_r} \otimes e^\beta \in V_M$; we define a degree operator d_0 of V_M by setting

$$d_0 v = \Big(-\sum_{i=1}^{r} i k_i - \frac{1}{2}(\beta, \beta) \Big) v. \tag{2.16}$$

If a is any non-zero complex number we define operators

$$z^\alpha . u \otimes e^\beta = z^{(\alpha,\beta)} u \otimes e^\beta, \quad a^\alpha . u \otimes e^\beta = a^{(\alpha,\beta)} u \otimes e^\beta \tag{2.17}$$

for $\alpha, \beta \in \Gamma_M$, $u \in \mathcal{S}(\mathcal{H}_M^-)$. Then a^α is just the evaluation map, at a, of the operator z^α. The following result is well-known.

LEMMA 2.18:

$$[d_0, E^{\pm}(\alpha, az)] = -D_z E^{\pm}(\alpha, az) = \Big(\sum_{k \in \mathbb{Z}_{\pm}} \alpha(k)(az)^{-k} \Big) E^{\pm}(\alpha, az),$$

$$[\alpha(0), z^\beta] = 0, \quad z^\alpha e^\beta = z^{(\alpha,\beta)} e^\beta z^\alpha$$

for $\alpha, \beta \in \Gamma_M$, $a \in \mathbb{C}^\times$, *where* $D_z = d/dz$.

We will need to raise some of the complex numbers which arise in our construction below to various powers and care must be taken with this. We thus set up the notation we use for this now. For any complex number $a \neq 0$, there is a unique real number $\theta \in [0, 2\pi)$ such that $a = |a| e^{\theta\sqrt{-1}}$. We define

$$Lna = \theta\sqrt{-1} + \ln |a|.$$

Viewing $\mathbb{C}^\times = \mathbb{C} \setminus \{0\}$ as multiplicative group, we call a subgroup G of $\mathbb{C}^\times$ **admissible** if $G = T \times F$, where $T = \langle \xi \rangle$ is a cyclic group of finite order $|T|$ and $F = \langle q_j | j \in J \rangle$ is a free abelian group with free generators $q_j, j \in J$. For $a =$

$\xi^{-n_0}q_{i_1}^{n_1}\cdots q_{i_k}^{n_k} \in G$, where $n_0, n_1, \ldots, n_k \in \mathbb{Z}, i_1, \ldots, i_k \in J$, $0 \le n_0 \le |T| - 1$, we define

$$a^r = e^{r(-n_0 Ln\xi + n_1 Lnq_{i_1} + \cdots + n_k Lnq_{i_k})} \tag{2.19}$$

for $r \in \mathbb{C}$.

Recall the definition of limit of formal power series from [FLM]. Let V be a vector space over $\mathbb{C}$. Let

$$f(z_1, z_2) = \sum_{i,j\in\mathbb{Z}} a_{ij} z_1^i z_2^j \in V[[z_1^{\pm 1}, z_2^{\pm 1}]];$$

we say the limit, $\lim_{z_2\to z_1} f(z_1, z_2)$, exists if, for any $l \in \mathbb{Z}$, $a_{i,l-i} = 0$ whenever $|i| >> 0$, and write

$$\lim_{z_2\to z_1} f(z_1, z_2) = f(z_1, z_1) = \sum_{l\in\mathbb{Z}} \Big(\sum_{i\in\mathbb{Z}} a_{i,l-i} \Big) z_1^l. \tag{2.20}$$

For our purposes we need another notion of limit as well. Let

$$f(x, z) = \sum_{i\in\mathbb{Z}} C_i(x) z^i,$$

where $C_i(x) = \sum_j c_{ij}(x) v_j \in \mathbb{C}(x) \otimes V$, and j runs over a finite set for each fixed $i \in \mathbb{Z}$, and $c_{ij}(x) \in \mathbb{C}(x)$ are complex rational functions. We say the limit, $\lim_{x\to a} f(x, z)$, exists if the function $c_{ij}(x)$ has a usual limit at the point $a \in \mathbb{C}$ for all $i, j \in \mathbb{Z}$, and write

$$\lim_{x\to a} f(x, z) = f(a, z) = \sum_i \Big(\sum_j c_{ij}(a) v_j \Big) z^j. \tag{2.21}$$

LEMMA 2.22: *For $1 \le i \le M$, we have*

$$\lim_{a\to 1} \frac{1}{1-a}(a^{-\varepsilon_i} - 1) = \varepsilon_i(0), \tag{2.23}$$

$$\lim_{a\to 1} \frac{1}{1-a}\Big(\sum_{k\in\mathbb{Z}_\pm} \frac{\varepsilon_i(k)}{k}(az)^{-k} - \sum_{k\in\mathbb{Z}_\pm} \frac{\varepsilon_i(k)}{k} z^{-k} \Big) = \sum_{k\in\mathbb{Z}_\pm} \varepsilon_i(k) z^{-k}. \tag{2.24}$$

Proof: For any $v = u \otimes e^\beta \in V_M$, let $m = -(\varepsilon_i, \beta) \in \mathbb{Z}$. Then

$$\frac{1}{1-a}(a^{-\varepsilon_i} - 1).v = \frac{a^m - 1}{1-a} v,$$

and

$$\lim_{a\to 1} \frac{1}{1-a}(a^{-\varepsilon_i} - 1).v = -mv = \varepsilon_i(0).v$$

as required. The second identity is clear. ∎

COROLLARY 2.25:

$$\lim_{a\to 1}\frac{1}{1-a}(E^{\pm}(\varepsilon_i,az)-E^{\pm}(\varepsilon_i,z))=E^{\pm}(\varepsilon_i,z)\sum_{k\in\mathbb{Z}_{\pm}}\varepsilon_i(k)z^{-k}.$$

Proof: Note that

$$E^{\pm}(\varepsilon_i,az)-E^{\pm}(\varepsilon_i,z)=\sum_{l=1}^{\infty}\frac{1}{l!}\left[\Big(\sum_{k\in\mathbb{Z}_{\pm}}\frac{\varepsilon_i(k)}{k}(az)^{-k}\Big)^{l}-\Big(\sum_{k\in\mathbb{Z}_{\pm}}\frac{\varepsilon_i(k)}{k}z^{-k}\Big)^{l}\right],$$

and $A^l-B^l=(A-B)\sum_{j=0}^{l-1}A^{l-1-j}B^j$; we obtain by applying the previous lemma

$$\lim_{a\to 1}\frac{1}{1-a}(E^{\pm}(\varepsilon_i,az)-E^{\pm}(\varepsilon_i,z))$$
$$=\sum_{l=1}^{\infty}\frac{1}{l!}\left[l\Big(\sum_{k\in\mathbb{Z}_{\pm}}\frac{\varepsilon_i(k)}{k}z^{-k}\Big)^{l-1}\right]\sum_{k\in\mathbb{Z}_{\pm}}\varepsilon_i(k)z^{-k}=E^{\pm}(\varepsilon_i,z)\sum_{k\in\mathbb{Z}_{\pm}}\varepsilon_i(k)z^{-k}.\qquad\blacksquare$$

COROLLARY 2.26:

$$\lim_{a\to 1}\frac{1}{1-a}(E^{\pm}(-\varepsilon_i,z)E^{\pm}(\varepsilon_i,az)-1)=\sum_{k\in\mathbb{Z}_{\pm}}\varepsilon_i(k)z^{-k}.$$

Proof: This follows from the fact that

$$E^{\pm}(-\varepsilon_i,z)E^{\pm}(\varepsilon_i,az)-1=E^{\pm}(-\varepsilon_i,z)(E^{\pm}(\varepsilon_i,az)-E^{\pm}(\varepsilon_i,z)),$$

and the previous corollary. ∎

For $\alpha\in\Gamma_M$, we define

$$X(\alpha,z)=E^{-}(-\alpha,z)E^{+}(-\alpha,z)e^{\alpha}z^{\alpha}z^{(\alpha,\alpha)/2}. \tag{2.27}$$

We may formally write

$$X(\alpha,z)=\sum_{k\in\mathbb{Z}+(\alpha,\alpha)/2}x_k(\alpha)z^{-k}, \tag{2.28}$$

where $x_k(\alpha)\in\mathrm{End}(V_M)$ for $k\in\mathbb{Z}+(\alpha,\alpha)/2$. It is known from [F1] that, if $(\alpha,\alpha)=1$, the operators $\{x_k(\alpha),x_k(-\alpha)|k\in\mathbb{Z}+\frac{1}{2}\}$ generate a Clifford algebra with the relations

$$\{x_k(\alpha),x_{-l}(-\alpha)\}=\delta_{kl},\ \{x_k(\alpha),x_l(\alpha)\}=0,\ \{x_k(-\alpha),x_l(-\alpha)\}=0 \tag{2.29}$$

for all $k,l\in\mathbb{Z}+\frac{1}{2}$. Related to this Clifford structure, we define the following normal ordering (see [F1] and [G3]):

$$:x_k(\varepsilon_i)x_{-l}(-\varepsilon_j):=x_k(\varepsilon_i)x_{-l}(-\varepsilon_j)-\delta_{ij}\delta_{kl}\theta(k) \tag{2.30}$$

for $k, l \in \mathbb{Z} + \frac{1}{2}$, $1 \leq i, j \leq M$, where $\theta(k) = 0$ if $k < 0$, $\theta(k) = 1$ if $k > 0$. By applying Lemmas 2.15 and 2.18, one can easily prove the following result:

LEMMA 2.31: *For $1 \leq i, j \leq M$, and $a \in \mathbb{C}^\times$, we have*

$$: X(\varepsilon_i, z_1) X(-\varepsilon_j, az_2) :$$
$$= \left(1 - \frac{az_2}{z_1}\right)^{-\delta_{ij}} \frac{(az_2)^{\delta_{ij}/2}}{z_1^{\delta_{ij}/2}} \Big(\epsilon(\varepsilon_i, \varepsilon_j) z_1^{(\varepsilon_i, \varepsilon_i - \varepsilon_j)/2} \cdot (az_2)^{-(\varepsilon_j, \varepsilon_i - \varepsilon_j)/2}$$
$$\cdot e^{\varepsilon_i - \varepsilon_j} z_1^{\varepsilon_i} (az_2)^{-\varepsilon_j} E^-(-\varepsilon_i, z_1) E^-(\varepsilon_j, az_2) E^+(-\varepsilon_i, z_1) E^+(\varepsilon_j, az_2) - \delta_{ij} \Big).$$

In particular, if $i \neq j$, then

$$(2.32) \quad : X(\varepsilon_i, z_1) X(-\varepsilon_j, az_2) := \epsilon(\varepsilon_i, \varepsilon_j) z_1^{1/2} (az_2)^{1/2} e^{\varepsilon_i - \varepsilon_j} z_1^{\varepsilon_i} (az_2)^{-\varepsilon_j} \cdot E^-(-\varepsilon_i, z_1) E^-(\varepsilon_j, az_2) E^+(-\varepsilon_i, z_1) E^+(\varepsilon_j, az_2),$$

and, if $i = j$, then

$$(2.33) \quad \left(1 - \frac{az_2}{z_1}\right) : X(\varepsilon_i, z_1) X(-\varepsilon_i, az_2) :$$
$$= \frac{(az_2)^{\frac{1}{2}}}{z_1^{\frac{1}{2}}} \left(\left(\frac{z_1}{az_2}\right)^{\varepsilon_i} E^-(-\varepsilon_i, z_1) E^-(\varepsilon_i, az_2) E^+(-\varepsilon_i, z_1) E^+(\varepsilon_i, az_2) - 1 \right).$$

PROPOSITION 2.34: *For $1 \leq i, j \leq M$, and $a \in \mathbb{C}^\times$, we have*

$$: X(\varepsilon_i, z) X(-\varepsilon_j, az) :$$
$$= \begin{cases} \epsilon(\varepsilon_i, \varepsilon_j) z^{\frac{1}{2}} (az)^{\frac{1}{2}} e^{\varepsilon_i - \varepsilon_j} z^{\varepsilon_i} (az)^{-\varepsilon_j} E^-(-\varepsilon_i, z) E^-(\varepsilon_j, az) E^+(-\varepsilon_i, z) \\ \qquad \times E^+(\varepsilon_j, az) \quad \text{if } i \neq j, \\ \varepsilon_i(z) \qquad \text{if } i = j, \quad a = 1, \\ \frac{a^{1/2}}{1-a} (a^{-\varepsilon_i} E^-(-\varepsilon_i, z) E^-(\varepsilon_i, az) E^+(-\varepsilon_i, z) E^+(\varepsilon_i, az) - 1) \\ \qquad \text{if } i = j, \quad a \neq 1. \end{cases}$$

Proof: Taking the limit $z_2 \to z_1$ in (2.32) and (2.33) gives the first and third identities. The second identity follows from Lemma 2.22, Corollary 2.26 and the third identity by taking the limit $a \to 1$. ∎

Remark 2.35: Note that the second identity in Proposition 2.34 was given in [F1].

Definition 2.36: For $a \in \mathbb{C}^\times$, $1 \leq i, j \leq M$, we define

$$X_{ij}(a, z) =: X(\varepsilon_i, z) X(-\varepsilon_j, az) :$$

Now we can state our main theorem of this section.

THEOREM 2.37: *For $a, b \in \mathbb{C}^\times$ and $1 \le i, j, k, l \le M$, we have:*
(i) *if $ab \neq 1$, then*

$$(2.38)\ [X_{ij}(a, z_1), X_{kl}(b, z_2)] = X_{il}(ab, z_1)\delta_{jk}\delta\Big(\frac{z_2}{az_1}\Big) - X_{kj}(ab, z_2)\delta_{il}\delta\Big(\frac{z_1}{bz_2}\Big) + \frac{a^{\frac12}b^{\frac12}}{1-ab}\delta_{il}\delta_{jk}\Big(\delta\Big(\frac{z_2}{az_1}\Big) - \delta\Big(\frac{z_1}{bz_2}\Big)\Big)c;$$

(ii) *if $ab = 1$, then*

$$(2.39)\ [X_{ij}(a, z_1), X_{kl}(b, z_2)] = (X_{il}(1, z_1)\delta_{jk} - X_{kj}(1, z_2)\delta_{il})\delta\Big(\frac{z_2}{az_1}\Big) + \delta_{il}\delta_{jk}(D\delta)\Big(\frac{z_2}{az_1}\Big)c.$$

The proof of Theorem 2.37 will be carried out in several steps. In what follows we will freely use the following two lemmas. Lemma 2.40 can be found in [FLM] and [K], and Lemma 2.43 can be found in [J], [G1, 2] or [BS].

LEMMA 2.40: *Let $Y(z_1, z_2)$ be a formal power series in z_1, z_2 with coefficients in a vector space, such that $\lim_{z_2 \to z_1} f(z_1, z_2)$ exists. Then*

$$(2.41)\qquad Y(z_1, z_2)\delta\Big(\frac{az_1}{z_2}\Big) = Y(z_1, az_1)\delta\Big(\frac{az_1}{z_2}\Big),$$

$$(2.42)\quad Y(z_1, z_2)(D\delta)\Big(\frac{z_2}{az_1}\Big) = Y(z_1, az_1)(D\delta)\Big(\frac{z_2}{az_1}\Big) - (D_{z_2}Y(z_1, z_2))\delta\Big(\frac{z_2}{az_1}\Big),$$

for $a \in \mathbb{C}^\times$.

LEMMA 2.43: *Suppose $a, b \in \mathbb{C}^\times$. Then*

$$\Big(1 - \frac{z_2}{az_1}\Big)^{-1}\Big(1 - \frac{bz_2}{z_1}\Big)^{-1} - \frac{az_1}{z_2}\frac{z_1}{bz_2}\Big(1 - \frac{z_1}{bz_2}\Big)^{-1}\Big(1 - \frac{az_1}{z_2}\Big)^{-1} = \begin{cases} (1-ab)^{-1}\frac{az_1}{z_2}\Big(\delta\Big(\frac{az_1}{z_2}\Big) - \delta\Big(\frac{z_1}{bz_2}\Big)\Big) & \text{if } ab \neq 1, \\ \frac{az_1}{z_2}(D\delta)\Big(\frac{z_2}{az_1}\Big) & \text{if } ab = 1. \end{cases}$$

Now we divide the proof for Theorem 2.37 into four different cases.

Case 1. $i \neq j$, $k \neq l$; **Case 2.** $i = j$, $k \neq l$, and $ab \neq 1$; **Case 3.** $i = j$, $k \neq l$, and $ab = 1$; **Case 4.** $i = j, k = l$.

We shall check the cases 2 and 4, and leave the other cases to the reader.

CASE 2: $i = j$, $k \neq l$, and $ab \neq 1$.

If $a = 1$, then

$$X_{ij}(a, z_1) = \varepsilon_i(z_1).$$

Since

$$[\varepsilon_i(z_1), e^{\varepsilon_k-\varepsilon_l}] = (\delta_{ik} - \delta_{il})e^{\varepsilon_k-\varepsilon_l},$$
$$[\varepsilon_i(z_1), E^{\pm}(-\varepsilon_k, z_2)] = \sum_{n\in\mathbb{Z}_\pm} \delta_{ik}\Big(\frac{z_1}{z_2}\Big)^n E^{\pm}(-\varepsilon_k, z_2),$$

and

$$[\varepsilon_i(z_1), E^{\pm}(\varepsilon_l, bz_2)] = -\sum_{n\in\mathbb{Z}_\pm} \delta_{il}\Big(\frac{z_1}{bz_2}\Big)^n E^{\pm}(\varepsilon_l, bz_2),$$

we have

$$\begin{aligned}[X_{ij}(a,z_1), X_{kl}(b,z_2)] &= [\varepsilon_i(z_1), X_{kl}(b,z_2)] = X_{kl}(b,z_2)\\ \cdot\Big(\delta_{ik}-\delta_{il}+\delta_{ik}\sum_{n\in\mathbb{Z}_-}\Big(\frac{z_1}{z_2}\Big)^n &+\delta_{ik}\sum_{n\in\mathbb{Z}_+}\Big(\frac{z_1}{z_2}\Big)^n-\delta_{il}\sum_{n\in\mathbb{Z}_-}\Big(\frac{z_1}{bz_2}\Big)^n-\delta_{il}\sum_{n\in\mathbb{Z}_+}\Big(\frac{z_1}{bz_2}\Big)^n\Big)\\ &=X_{kl}(b,z_2)\Big(\delta_{ik}\delta\Big(\frac{z_2}{z_1}\Big)-\delta_{il}\delta\Big(\frac{z_1}{bz_2}\Big)\Big)\\ &=X_{kl}(b,z_1)\delta_{jk}\delta\Big(\frac{z_2}{z_1}\Big)-X_{kj}(b,z_2)\delta_{il}\delta\Big(\frac{z_1}{bz_2}\Big),\end{aligned}$$

as needed.

If $a \neq 1$, then

$$X_{ij}(a,z_1) = \frac{a^{1/2}}{1-a}(a^{-\varepsilon_i}E^-(-\varepsilon_i,z_1)E^-(\varepsilon_i,az_1)E^+(-\varepsilon_i,z_1)E^+(\varepsilon_i,az_1)-1).$$

Applying Lemmas 2.15 and 2.18, we get

$$\begin{aligned}(2.45)\qquad &[X_{ij}(a,z_1), X_{kl}(b,z_2)]\\ =&\frac{a^{1/2}}{1-a}b^{\frac{1}{2}}\epsilon(\varepsilon_k,\varepsilon_l)e^{\varepsilon_k-\varepsilon_l}a^{-\varepsilon_i}b^{-\varepsilon_l}z_2^{\varepsilon_k-\varepsilon_l}z_2\\ &\cdot E^-(-\varepsilon_i,z_1)E^-(\varepsilon_i,az_1)E^-(-\varepsilon_k,z_2)E^-(\varepsilon_l,bz_2)\\ &\cdot E^+(-\varepsilon_i,z_1)E^+(\varepsilon_i,az_1)E^+(-\varepsilon_k,z_2)E^+(\varepsilon_l,bz_2)Q(z_1,z_2),\end{aligned}$$

where

$$\begin{aligned}
Q(z_1,z_2) =&a^{-(\varepsilon_j,\varepsilon_k-\varepsilon_l)}\Big(1-\frac{z_2}{z_1}\Big)^{\delta_{ik}}\Big(1-\frac{z_2}{az_1}\Big)^{-\delta_{ik}}\Big(1-\frac{bz_2}{z_1}\Big)^{-\delta_{il}}\Big(1-\frac{bz_2}{az_1}\Big)^{\delta_{il}}\\
&-\Big(1-\frac{z_1}{z_2}\Big)^{\delta_{ik}}\Big(1-\frac{z_1}{bz_2}\Big)^{-\delta_{il}}\Big(1-\frac{az_1}{z_2}\Big)^{-\delta_{ik}}\Big(1-\frac{az_1}{bz_2}\Big)^{\delta_{il}}\\
=&a^{\delta_{il}-\delta_{ik}}\Big(1-\frac{z_2}{z_1}\Big)^{\delta_{ik}}\Big(1-\frac{bz_2}{az_1}\Big)^{\delta_{il}}\Big(\Big(1-\frac{z_2}{az_1}\Big)^{-\delta_{ik}}\Big(1-\frac{bz_2}{z_1}\Big)^{-\delta_{il}}\\
&-(-1)^{\delta_{ik}+\delta_{il}}\Big(\frac{az_1}{z_2}\Big)^{\delta_{ik}}\Big(\frac{z_1}{bz_2}\Big)^{\delta_{il}}\Big(1-\frac{z_1}{bz_2}\Big)^{-\delta_{il}}\Big(1-\frac{az_1}{z_2}\Big)^{-\delta_{ik}}\Big)\\
=&\begin{cases}0 & \text{if } i\neq k, i\neq l,\\ \frac{1-a}{a}\delta\Big(\frac{z_2}{az_1}\Big) & \text{if } i=k, i\neq l,\\ (a-1)\delta\Big(\frac{z_1}{bz_2}\Big) & \text{if } i\neq k, i=l.\end{cases}
\end{aligned}$$

We thus have (2.38) by applying Lemma 2.40.

CASE 4: $i=j, k=l$.

Note that

$$X_{ij}(a,z_1)$$
$$=\begin{cases}\frac{a^{1/2}}{1-a}(a^{-\varepsilon_i}E^-(-\varepsilon_i,z_1)E^-(\varepsilon_i,az_1)E^+(-\varepsilon_i,z_1)E^+(\varepsilon_i,az_1)-1) & \text{if } i=j, a\neq 1,\\ \varepsilon_i(z_1) & \text{if } i=j, a=1;\end{cases}$$
$$X_{kl}(b,z_2)$$
$$=\begin{cases}\frac{b^{1/2}}{1-b}(b^{-\varepsilon_k}E^-(-\varepsilon_k,z_2)E^-(\varepsilon_k,bz_2)E^+(-\varepsilon_k,z_2)E^+(\varepsilon_k,bz_2)-1) & \text{if } k=l, b\neq 1,\\ \varepsilon_k(z_2) & \text{if } k=l, b=1.\end{cases}$$

We consider three subcases. First we assume that $a=b=1$; then

$$[X_{ij}(a,z_1),X_{kl}(b,z_2)]=[\varepsilon_i(z_1),\varepsilon_k(z_2)]=\delta_{ik}(D\delta)\Big(\frac{z_2}{z_1}\Big),$$

as desired.

Next we assume $a=1$, $b\neq 1$; then

$$\begin{aligned}
&[X_{ij}(a,z_1),X_{kl}(b,z_2)]=[\varepsilon_i(z_1),X_{kl}(b,z_2)]\\
&=\frac{b^{1/2}}{1-b}b^{-\varepsilon_k}E^-(-\varepsilon_k,z_2)E^-(\varepsilon_k,bz_2)E^+(-\varepsilon_k,z_2)E^+(\varepsilon_k,bz_2)\\
&\cdot\Big(\delta_{ik}\sum_{n\in\mathbb{Z}_-}\Big(\frac{z_2}{z_1}\Big)^n+\delta_{ik}\sum_{n\in\mathbb{Z}_+}\Big(\frac{z_2}{z_1}\Big)^n-\delta_{ik}\sum_{n\in\mathbb{Z}_-}\Big(\frac{z_1}{bz_2}\Big)^n-\delta_{ik}\sum_{n\in\mathbb{Z}_+}\Big(\frac{z_1}{bz_2}\Big)^n\Big)\\
=&\frac{b^{1/2}}{1-b}b^{-\varepsilon_k}E^-(-\varepsilon_k,z_2)E^-(\varepsilon_k,bz_2)E^+(-\varepsilon_k,z_2)E^+(\varepsilon_k,bz_2)\delta_{ik}\Big(\delta\Big(\frac{z_2}{z_1}\Big)-\delta\Big(\frac{z_1}{bz_2}\Big)\Big)\\
=&X_{il}(ab,z_1)\delta_{jk}\delta\Big(\frac{z_2}{az_1}\Big)-X_{kj}(ab,z_2)\delta_{il}\delta\Big(\frac{z_1}{bz_2}\Big)+\frac{a^{\frac12}b^{\frac12}}{1-ab}\delta_{il}\delta_{jk}\Big(\delta(\frac{z_2}{az_1})-\delta\Big(\frac{z_1}{bz_2}\Big)\Big),
\end{aligned}$$

as required.

Finally, we assume $a \neq 1$, $b \neq 1$; then

$$\begin{aligned}(2.47)\qquad & [X_{ij}(a,z_1),X_{kl}(b,z_2)] \\ &= \frac{a^{1/2}}{1-a}\frac{b^{1/2}}{1-b}a^{-\varepsilon_i}b^{-\varepsilon_k}E^-(-\varepsilon_i,z_1)E^-(-\varepsilon_k,z_2)E^-(\varepsilon_i,az_1) \\ &\cdot E^-(\varepsilon_k,bz_2)E^+(-\varepsilon_i,z_1)E^+(-\varepsilon_k,z_2)E^+(\varepsilon_i,az_1)E^+(\varepsilon_k,bz_2S(z_1,z_2),\end{aligned}$$

where

$$\begin{aligned} S(z_1,z_2) =& \Big(1-\frac{z_2}{z_1}\Big)^{\delta_{ik}}\Big(1-\frac{bz_2}{z_1}\Big)^{-\delta_{ik}}\Big(1-\frac{z_2}{az_1}\Big)^{-\delta_{ik}}\Big(1-\frac{bz_2}{az_1}\Big)^{\delta_{ik}} \\ &-\Big(1-\frac{z_1}{z_2}\Big)^{\delta_{ik}}\Big(1-\frac{z_1}{bz_2}\Big)^{-\delta_{ik}}\Big(1-\frac{az_1}{z_2}\Big)^{-\delta_{ik}}\Big(1-\frac{az_1}{bz_2}\Big)^{\delta_{ik}} \\ =&\Big(1-\frac{z_2}{z_1}\Big)^{\delta_{ik}}\Big(1-\frac{bz_2}{az_1}\Big)^{\delta_{ik}}\bigg(\Big(1-\frac{bz_2}{z_1}\Big)^{-\delta_{ik}}\Big(1-\frac{z_2}{az_1}\Big)^{-\delta_{ik}} \\ &-\Big(\frac{az_1}{z_2}\Big)^{\delta_{ik}}\Big(\frac{z_1}{bz_2}\Big)^{\delta_{ik}}\Big(1-\frac{z_1}{bz_2}\Big)^{-\delta_{ik}}\Big(1-\frac{az_1}{z_2}\Big)^{-\delta_{ik}}\bigg) \\ =&\begin{cases}\Big(1-\frac{z_2}{z_1}\Big)\Big(1-\frac{bz_2}{az_1}\Big)\frac{az_1}{z_2}(D\delta)\Big(\frac{z_2}{az_1}\Big) & \text{if } i=k, ab=1, \\ 0 & \text{if } i\neq k, \\ \frac{1}{1-ab}\Big(1-\frac{z_2}{z_1}\Big)\Big(1-\frac{bz_2}{az_1}\Big)\frac{az_1}{z_2}\Big(\delta\Big(\frac{z_2}{az_1}\Big)-\delta\Big(\frac{z_1}{bz_2}\Big)\Big) & \text{if } i=k, ab\neq 1.\end{cases}\end{aligned}$$

Substitute $S(z_1,z_2)$ back into (2.47) and apply Lemma 2.40 to get (2.38) and (2.39). This now completes the proof of Theorem 2.37.

Remark 2.48: If $ab \neq 1$, we have

$$\delta\Big(\frac{z_2}{az_1}\Big)-\delta\Big(\frac{z_1}{bz_2}\Big)=\sum_{m\in\mathbb{Z}}(1-(ab)^m)\Big(\frac{z_2}{az_1}\Big)^m,$$

thus

$$\begin{aligned}&\frac{a^{\frac12}b^{\frac12}}{1-ab}\delta_{il}\delta_{jk}\Big(\delta\Big(\frac{z_2}{az_1}\Big)-\delta\Big(\frac{z_1}{bz_2}\Big)\Big)=a^{\frac12}b^{\frac12}\delta_{il}\delta_{jk}\sum_{m\in\mathbb{Z}}\frac{1-(ab)^m}{1-ab}\Big(\frac{z_2}{az_1}\Big)^m \\ &=a^{\frac12}b^{\frac12}\delta_{il}\delta_{jk}\bigg(\sum_{m\in\mathbb{Z}_+}\Big(\sum_{s=0}^{m-1}(ab)^s\Big)(\frac{z_2}{az_1})^m+\sum_{m\in\mathbb{Z}_-}\Big(-\sum_{s=1}^{-m}(ab)^{-s}\Big)\Big(\frac{z_2}{az_1}\Big)^m\bigg),\end{aligned}$$

which gives

$$\begin{aligned}&\lim_{b\to a^{-1}}\frac{a^{\frac12}b^{\frac12}}{1-ab}\delta_{il}\delta_{jk}\Big(\delta\Big(\frac{z_2}{az_1}\Big)-\delta\Big(\frac{z_1}{bz_2}\Big)\Big) \\ &=\delta_{il}\delta_{jk}\bigg(\sum_{m\in\mathbb{Z}_+}m\Big(\frac{z_2}{az_1}\Big)^m+\sum_{m\in\mathbb{Z}_-}m\Big(\frac{z_2}{az_1}\Big)^m\bigg)=\delta_{il}\delta_{jk}(D\delta)\Big(\frac{z_2}{az_1}\Big).\end{aligned}$$

This indicates that the second identity of Theorem 2.37 can be obtained from the first one by taking the limit as $b \to a^{-1}$.

3. Lie algebras and representations

In this section we are going to define a class of Lie algebras from our vertex operators which will correspond to admissible subgroups of $\mathbb{C}^{\times}$. Indeed, for some choice of the admissible group G and positive integer M, the Lie algebra $\mathcal{G}(G, M)$ (defined below) of operators, which act on the Fock space V_M, will give realizations of some infinite-dimensional Lie algebras studied in Section 1. This will include the affine algebra $\widehat{gl_M}(\mathbb{C})$ in both the principal and homogeneous pictures as well as some Lie algebras with quantum torus coordinates. Towards this end, we first introduce some new notations for the vertex operators constructed in the preceding section.

Definition 3.1: For $a, b \in \mathbb{C}^{\times}$, $1 \le i, j \le M$, we set $X_{ij}(a, b, z) := X_{ij}(a^{-1}b, az)$, and write

$$X_{ij}(a, b, z) = \sum_{k \in \mathbb{Z}} x_{ij}(k, a, b) z^{-k} \tag{3.2}$$

where $x_{ij}(k, a, b) \in \mathrm{End} V_M$.

With this notation Theorem 2.37 can be re-written as follows:

THEOREM 3.3: *Let $a_1, a_2, b_1, b_2 \in \mathbb{C}^{\times}$, and $1 \le i, j, k, l \le M$. We have*

(i) *if $a_1 a_2 \neq b_1 b_2$, then*

$$\begin{aligned}
&[X_{ij}(a_1, b_1, z_1), X_{kl}(a_2, b_2, z_2)] \\
=& X_{il}\Big(a_1, \frac{b_1 b_2}{a_2}, z_1\Big)\delta_{jk}\delta\Big(\frac{a_2 z_2}{b_1 z_1}\Big) - X_{kj}\Big(a_2, \frac{b_1 b_2}{a_1}, z_2\Big)\delta_{il}\delta\Big(\frac{a_1 z_1}{b_2 z_2}\Big) \\
&+ \frac{(a_1^{-1}b_1)^{\frac{1}{2}}(a_2^{-1}b_2)^{\frac{1}{2}}}{1 - a_1^{-1}b_1 a_2^{-1}b_2}\delta_{il}\delta_{jk}\Big(\delta\Big(\frac{a_2 z_2}{b_1 z_1}\Big) - \delta\Big(\frac{a_1 z_1}{b_2 z_2}\Big)\Big)c;
\end{aligned}$$

(ii) *if $a_1 a_2 = b_1 b_2$, then*

$$\begin{aligned}
&[X_{ij}(a_1, b_1, z_1), X_{kl}(a_2, b_2, z_2)] \\
=& X_{il}\Big(a_1, \frac{b_1 b_2}{a_2}, z_1\Big)\delta_{jk}\delta\Big(\frac{a_2 z_2}{b_1 z_1}\Big) - X_{kj}\Big(a_2, \frac{b_1 b_2}{a_1}, z_2\Big)\delta_{il}\delta\Big(\frac{a_1 z_1}{b_2 z_2}\Big) \\
&+ \delta_{il}\delta_{jk}(D\delta)\Big(\frac{a_2 z_2}{b_1 z_1}\Big)c.
\end{aligned}$$

Fix an integer $M \ge 1$ and an admissible subgroup G of $\mathbb{C}^{\times}$. Let $\mathcal{G}(G, M)$ be the vector space spanned by c and all of the coefficients of the vertex operators $X_{ij}(a, b, z)$ for all $1 \le i, j \le M$, and $a, b \in G$. Then we have the following result.

THEOREM 3.4: *$\mathcal{G}(G,M)$ forms a Lie algebra of operators acting on the Fock space V_M. Moreover,*

$$V_M = \bigoplus_{k\in\mathbb{Z}} V_M^{(k)}$$

where $V_M^{(k)} = e^{k\varepsilon_M + Q_M} \otimes \mathcal{S}(\mathcal{H}_M^-)$, and $V_M^{(k)}$ is an irreducible $\mathcal{G}(G,M)$-module.

Proof: It is obvious from Theorem 3.3 that $\mathcal{G}(G,M)$ is a Lie algebra and that $V_M^{(k)}$ is a $\mathcal{G}(G,M)$-module. To see it is irreducible, we note that the Heisenberg algebra $\widehat{\mathcal{H}}_M \subset \mathcal{H}_M$, and $\mathcal{H}_M$ is spanned by the coefficient operators of the vertex operators $X_{ii}(1,1,z)$ for $1 \le i \le M$. This then implies that, if W is a non-zero submodule of $V_M^{(k)}$, we can choose a non-zero element of the form $v = e^{k\varepsilon_M+\alpha} \otimes 1 \in W$ for some $\alpha \in Q_M$. Moreover, it is easy to check that

$$x_{ij}(n_{ij}-1,1,1).v = \epsilon(\varepsilon_i,\varepsilon_j)\epsilon(\varepsilon_i-\varepsilon_j, k\varepsilon_M+\alpha)e^{k\varepsilon_M+\alpha+\varepsilon_i-\varepsilon_j}$$

for all $1 \le i \ne j \le M$, where $n_{ij} = (\varepsilon_j - \varepsilon_i, k\varepsilon_M + \alpha) \in \mathbb{Z}$. Therefore $e^{k\varepsilon_M+\beta} \otimes 1 \in W$ for any $\beta \in Q_M$. This thus gives $W = V_M^{(k)}$ as needed. ∎

Remark 3.5: Note that the coefficients of the vertex operators $X_{ij}(a,z)$ and $X_{ij}(a,bz)$ for any given $a,b \in G$ span the same space. Thus $\mathcal{G}(G,M)$ is spanned by c and the coefficients of the operators $X_{ij}(a,z)$ for $1 \le i,j \le M, a \in G$. Therefore, it follows from Theorem 4.25 in [G1] that $\mathcal{G}(G,M)$ is an affinization of the Lie algebra $gl_M(\mathcal{R}[t,t^{-1};\tau])$, where $\mathcal{R} = \mathbb{C}[G]$ is the group algebra and $\mathcal{R}[t,t^{-1};\tau]$ is the skew Laurent polynomial ring.

Recall definition (2.19). We extend the cocycle map ϵ: $H_M \times H_M \to \{\mathbb{C}^\times\}$ by defining

$$\epsilon\Big(\sum r_i\varepsilon_i, \sum s_i\varepsilon_i\Big) = \prod_{i,j}(\epsilon(\varepsilon_i,\varepsilon_j))^{r_is_j} \tag{3.6}$$

for $r_i, s_i \in \mathbb{C}$. It is obvious that

$$\epsilon(\alpha+\beta,\gamma) = \epsilon(\alpha,\gamma)\epsilon(\beta,\gamma),\ \epsilon(\alpha,\beta+\gamma) = \epsilon(\alpha,\beta)\epsilon(\alpha,\gamma)$$

for $\alpha,\beta,\gamma \in H_M$. Moreover, if we restrict ϵ to $\Gamma_M \times \Gamma_M$, then ϵ gives us the 2-cocycle defined in the previous section.

Let $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$, $i = 1,\ldots,M-1$, and $\alpha_M = \varepsilon_1 + \cdots + \varepsilon_M$. Then $Q_M = \bigoplus_{i=1}^{M-1}\mathbb{Z}\alpha_i$ and $\Gamma_M = \bigoplus_{i=1}^{M}\mathbb{Z}\alpha_i$. Let $Q_M^0 = \{\alpha \in \mathbb{C}\otimes_{\mathbb{Z}} Q_M | (\alpha, Q_M) \in \mathbb{Z}\}$ be the dual of the lattice Q_M and set

$$L_M^0 = \{\alpha \in H_M = \mathbb{C}\otimes_{\mathbb{Z}}\Gamma_M | (\alpha, Q_M) \in \mathbb{Z}\}. \tag{3.7}$$

Then $L_M^0 = Q_M^0 \oplus \mathbb{C}\alpha_M$. Let $I = L_M^0/Q_M$ then we have a Q_M-coset decomposition of L_M^0,

$$L_M^0 = \bigoplus_{i \in I} (\lambda_i + Q_M), \tag{3.8}$$

for some $\lambda_i \in L_M^0$.

PROPOSITION 3.9: *The Lie algebra of operators, $\mathcal{G}(G, M)$, acts on the space $V_M^{(\lambda_i)} = \mathcal{S}(\mathcal{H}_M^-) \otimes \mathbb{C}[Q_M + \lambda_i]$, and $V_M^{(\lambda_i)}$ affords an irreducible representation of $\mathcal{G}(G, M)$ for $i \in I = L_M^0/Q_M$. Moreover, $V_M^{(\lambda_i)} \cong V_M^{(\lambda_j)}$ if and only if $i = j$.*

Proof: For $u \otimes e^{\alpha+\lambda_i} \in V_M^{(\lambda_i)}$, $u \in \mathcal{S}(\mathcal{H}_M^-)$, $\alpha \in Q_M$, we note that

$$z^\beta.(u \otimes e^{\alpha+\lambda_i}) = z^{(\beta,\alpha+\lambda_i)} u \otimes e^{\alpha+\lambda_i}, a^\gamma.(u \otimes e^{\alpha+\lambda_i}) = a^{(\gamma,\alpha+\lambda_i)} u \otimes e^{\alpha+\lambda_i},$$
$$e^\beta.(u \otimes e^{\alpha+\lambda_i}) = \epsilon(\beta, \alpha + \lambda_i) u \otimes e^{\alpha+\beta+\lambda_i}$$

where $(\beta, \alpha + \lambda_i) \in \mathbb{Z}$, $(\gamma, \alpha + \lambda_i) \in \mathbb{C}$, $\epsilon(\beta, \alpha + \lambda_i) \in \mathbb{C}^\times$ for $\beta \in Q_M$, $\gamma \in \Gamma_M$ and $a \in G$. This implies that $X_{ij}(a, b, z).(u \otimes e^{\alpha+\lambda_i}) \in V_M^{(\lambda_i)}[[z, z^{-1}]]$, and so the Lie algebra $\mathcal{G}(G, M)$ acts on the space $V_M^{(\lambda_i)}$. The irreducibility of $V_M^{(\lambda_i)}$, for $i \in I$, follows from a similar argument as in Theorem 3.4. The last part of this proposition is clear. ∎

4. Applications

In this section we assume the admissible subgroup G has the form $G = T \times F \subset \mathbb{C}^\times$, where $T = \langle \xi \rangle$ is generated by a root of unity ξ, and F is a free group with a finite number of generators. First, let $G = \{1\}$ and let $M \geq 2$ be any integer. Then the Lie algebra $\mathcal{G}(G, M)$ is generated by the coefficients of the vertex operators $X_{ij}(1, 1, z)$ for all $1 \leq i, j \leq M$. Moreover, from Theorem 3.3, we see that

$$\begin{aligned}[X_{ij}(1,1,z_1), X_{kl}(1,1,z_2)] =& X_{il}(1,1,z_1)\delta_{jk}\delta\Big(\frac{z_2}{z_1}\Big) - X_{kj}(1,1,z_2)\delta_{il}\delta\Big(\frac{z_1}{z_2}\Big) \\ &+ \delta_{il}\delta_{jk}(D\delta)\Big(\frac{z_2}{z_1}\Big)\end{aligned} \tag{4.1}$$

for $1 \leq i, j \leq M$. Comparing (4.1) with (1.17), we obtain the following result which was originally due to [F1]; see also [FK] and [S].

COROLLARY 4.2: *Let $G = \{1\}$ and $M \geq 2$. Then $\mathcal{G}(G, M)$ gives a representation of the affine algebra $\widehat{gl_M}(\mathbb{C})$ in the homogeneous picture on the Fock space V_M,* *and the representation is given by the mapping:*

$$E_{ij} \otimes t_0^k \mapsto x_{ij}(k,1,1),$$
$$c_0 \mapsto c$$

for $1 \leq i,j \leq M$ *and* $k \in \mathbb{Z}$.

Next we choose G to be a cyclic group of order $N \geq 2$ with generator $\xi = \xi_N$, and take $M = 1$. Note that

$$X_{11}(\xi^i, \xi^j, z) = X_{11}(\xi^{i-j-1}, \xi^{-1}, \xi^{j+1} z)$$

for $0 \leq i,j \leq N-1$. This implies that the Lie algebra $\mathcal{G}(G,M)$ is generated by the coefficients of the vertex operators $X_{11}(\xi^{i-1}, \xi^{-1}, z)$ for $0 \leq i \leq N-1$. From Theorem 3.3 we have

$$[X_{11}(\xi^{i-1},\xi^{-1},z_1), X_{11}(\xi^{j-1},\xi^{-1},z_2)]$$
$$= \begin{cases} X_{11}(\xi^{i-1},\xi^{-j-1},z_1)\delta\Big(\frac{\xi^j z_2}{z_1}\Big) - X_{11}(\xi^{j-1},\xi^{-i-1},z_2)\delta\Big(\frac{\xi^i z_1}{z_2}\Big) + (D\delta)\Big(\frac{\xi^j z_2}{z_1}\Big)c \\ \text{if } i+j = 0 (\operatorname{mod} N) \\ X_{11}(\xi^{i-1},\xi^{-j-1},z_1)\delta\Big(\frac{\xi^j z_2}{z_1}\Big) - X_{11}(\xi^{j-1},\xi^{-i-1},z_2)\delta\Big(\frac{\xi^i z_1}{z_2}\Big) \\ +\frac{e^{-\frac{i+j}{2}Ln\xi}}{\xi^{i+j}-1}\Big(\delta\Big(\frac{\xi^j z_2}{z_1}\Big) - \delta\Big(\frac{\xi^i z_1}{z_2}\Big)\Big) \text{if } i+j \neq 0 \ (\operatorname{mod} N). \end{cases}$$

Recalling the definition of $X_{ij}(a,b,z)$, we have

$$\begin{aligned} X_{11}(\xi^{i-1},\xi^{-j-1},z_1)\delta\Big(\frac{\xi^j z_2}{z_1}\Big) &= X_{11}(\xi^{i-1},\xi^{-j-1},\xi^j z_2)\delta\Big(\frac{\xi^j z_2}{z_1}\Big) \\ &= X_{11}(\xi^{i+j-1},\xi^{-1},z_2)\delta\Big(\frac{\xi^j z_2}{z_1}\Big), \end{aligned}$$

while

$$\begin{aligned} X_{11}(\xi^{j-1},\xi^{-i-1},z_2)\delta\Big(\frac{\xi^i z_1}{z_2}\Big) &= X_{11}(\xi^{i-1},\xi^{-j-1},\xi^j z_1)\delta\Big(\frac{\xi^i z_1}{z_2}\Big) \\ &= X_{11}(\xi^{i+j-1},\xi^{-1},z_1)\delta\Big(\frac{\xi^i z_1}{z_2}\Big) \end{aligned}$$

for $0 \leq i,j \leq N-1$. Therefore we get

$$\text{(4.3)} \qquad [X_{11}(\xi^{i-1},\xi^{-1},z_1), X_{11}(\xi^{j-1},\xi^{-1},z_2)]$$
$$= \begin{cases} X_{11}(\xi^{i+j-1},\xi^{-1},z_2)\delta\Big(\frac{\xi^j z_2}{z_1}\Big) - X_{11}(\xi^{i+j-1},\xi^{-1},z_1)\delta\Big(\frac{\xi^i z_1}{z_2}\Big) + (D\delta)\Big(\frac{\xi^j z_2}{z_1}\Big)c \\ \text{if } i+j = 0 \ (\operatorname{mod} N) \\ X_{11}(\xi^{i+j-1},\xi^{-1},z_2)\delta\Big(\frac{\xi^j z_2}{z_1}\Big) - X_{11}(\xi^{i+j-1},\xi^{-1},z_1)\delta\Big(\frac{\xi^i z_1}{z_2}\Big) \\ +\frac{e^{-\frac{i+j}{2}Ln\xi}}{\xi^{i+j}-1}\Big(\delta\Big(\frac{\xi^j z_2}{z_1}\Big) - \delta\Big(\frac{\xi^i z_1}{z_2}\Big)\Big) \text{if } i+j \neq 0 \ (\operatorname{mod} N)c. \end{cases}$$

Comparing this with the identity (1.18), we obtain the following result which was originally due to [F1] and [KKLW].

Corollary 4.4: *Let $M = 1$, and let G be the group generated by ξ, where ξ is an N-th primitive root of unity for $N \geq 2$. Then the Lie algebra $\mathcal{G}(G, M)$ gives a representation of the affine algebra $\widehat{gl_N}(\mathbb{C})$ on the Fock space V_1 in the principal picture. The representation is given by the mapping*

$$F^i E^k \otimes t_0^k \mapsto \begin{cases} x_{11}(k, \xi^{i-1}, \xi^{-1}) + \frac{e^{\frac{i}{2}Ln\xi}}{\xi^i - 1}\delta_{k0}c & \textit{if } 1 \leq i \leq N-1, \\ x_{11}(k, \xi^{-1}, \xi^{-1}) & \textit{if } i = 0, \end{cases}$$
$$c_0 \mapsto \frac{c}{N}$$

for $k \in \mathbb{Z}$.

Next we choose $M \geq 2$ and $G = \langle q \rangle$ where $q \neq 0$ is not a root of unity. Note that $X_{ij}(a, b, z) = X_{ij}(1, a^{-1}b, az)$ for $a, b \in G$. We see that the Lie algebra $\mathcal{G}(G, M)$ is generated by the coefficients of the vertex operators of the form $X_{ij}(1, q^r, z)$ for all $r \in \mathbb{Z}$ and $1 \leq i, j \leq M$. We apply Theorem 3.3 to obtain:

(i) if $r + s = 0$, then

$$\begin{aligned} (4.5) \qquad & [X_{ij}(1, q^r, z_1), X_{kl}(1, q^s, z_2)] \\ = X_{il}(1, q^{r+s}, z_1)\delta_{jk}\delta\Big(\frac{z_2}{q^r z_1}\Big) &- X_{kj}(1, q^{r+s}, z_2)\delta_{il}\delta\Big(\frac{z_1}{q^s z_2}\Big) + \delta_{il}\delta_{jk}(D\delta)\Big(\frac{z_2}{q^r z_1}\Big)c; \end{aligned}$$

(ii) if $r + s \neq 0$, then

$$\begin{aligned} (4.6) \qquad & [X_{ij}(1, q^r, z_1), X_{kl}(1, q^s, z_2)] \\ &= X_{il}(1, q^{r+s}, z_1)\delta_{jk}\delta\Big(\frac{z_2}{q^r z_1}\Big) - X_{kj}(1, q^{r+s}, z_2)\delta_{il}\delta\Big(\frac{z_1}{q^s z_2}\Big) \\ &+ \frac{q^{\frac{r+s}{2}}}{1 - q^{r+s}}\delta_{il}\delta_{jk}\Big(\delta(\frac{z_2}{q^r z_1}) - \delta\Big(\frac{z_1}{q^s z_2}\Big)\Big)c. \end{aligned}$$

Comparing the above two identities with the identity (1.19), we derive the following result which was given in [G1].

Corollary 4.7: *Let $M \geq 2$, and let $G = \langle q \rangle$ be the group generated by $q \neq 0$ where q is not a root of unity. Then the Lie algebra $\mathcal{G}(G, M)$ of operators, acting on V_M, gives a representation of the Lie algebra $\widehat{gl_M}(\mathbb{C}_q)$. The representation is given by the mapping*

$$E_{ij} \otimes t_0^m t_1^r \mapsto \begin{cases} x_{ij}(m, 1, q^r) + \frac{q^{r/2}}{1-q^r}\delta_{ij}\delta_{m0}c & \textit{if } r \neq 0, \\ x_{ij}(m, 1, 1) & \textit{if } r = 0, \end{cases}$$
$$c_0 \mapsto c, c_1 \mapsto 0$$

for $m, r \in \mathbb{Z}$.

Remark 4.8: The representation of the Lie algebra $\widehat{gl_M}(\mathbb{C}_Q)$ given in Corollary 4.7 is called the homogeneous realization. This is because of the fact that we are using the homogeneous gradation. Moreover, the algebra $\mathcal{G}(G, M)$ contains a subalgebra of $\mathcal{G}(\langle 1\rangle, M)$ which is generated by the operators $x_{ij}(m,1,1)$ for $1 \le i, j \le M$ and $m \in \mathbb{Z}$, and it is clear that this subalgebra is nothing but the affine algebra $\widehat{gl_M}(\mathbb{C})$ in the homogeneous picture.

Similarly, we may have the principal realization of the Lie algebra $\widehat{gl_N}(\mathbb{C}_Q)$. For this purpose, we choose the group $G = \langle \xi, q\rangle$ where $q \ne 0$ is not a root of unity and ξ is a N-th primitive root of unity. Let $M = 1$. Then the Lie algebra $\mathcal{G}(G, M)$ is generated by the coefficients of the vertex operators of the form $X_{11}(\xi^{i-1}, \xi^{-1}q^r, z)$ for all $r \in \mathbb{Z}$ and $0 \le i \le N-1$. From Theorem 3.3 we have:

(i) if $r + s = 0$ and $\overline{i+j} = 0 \pmod N$, then

$$\begin{aligned}&[X_{11}(\xi^{i-1}, \xi^{-1}q^r, z_1), X_{11}(\xi^{j-1}, \xi^{-1}q^s, z_2)]\\ =&X_{11}(\xi^{i+j-1}, \xi^{-1}q^{r+s}, \xi^{-j}z_1)\delta\Big(\frac{\xi^j z_2}{q^r z_1}\Big)\\ &- X_{11}(\xi^{i+j-1}, \xi^{-1}q^{r+s}, \xi^{-i}z_2)\delta\Big(\frac{\xi^i z_1}{q^s z_2}\Big) + (D\delta)\Big(\frac{\xi^j z_2}{q^r z_1}\Big)c;\end{aligned}$$

(ii) if $r + s \ne 0$ or $\overline{i+j} \ne 0 \pmod N$, then

$$\begin{aligned}&[X_{11}(\xi^{i-1}, \xi^{-1}q^r, z_1), X_{11}(\xi^{j-1}, \xi^{-1}q^s, z_2)]\\ =&X_{11}(\xi^{i+j-1}, \xi^{-1}q^{r+s}, \xi^{-j}z_1)\delta\Big(\frac{\xi^j z_2}{q^r z_1}\Big)\\ &- X_{11}(\xi^{i+j-1}, \xi^{-1}q^{r+s}, \xi^{-i}z_2)\delta\Big(\frac{\xi^i z_1}{q^s z_2}\Big)\\ &+ \frac{e^{\frac{i+j}{2}Ln\xi}q^{\frac{r+s}{2}}}{\xi^{i+j} - q^{r+s}}\Big(\delta\Big(\frac{\xi^j z_2}{q^r z_1}\Big) - \delta\Big(\frac{\xi^i z_1}{q^s z_2}\Big)\Big)c.\end{aligned}$$

Thus if we write

$$\bar{X}_{11}(\xi^{i-1}, \xi^{-1}q^r, z) = \begin{cases} X_{11}(\xi^{-1}, \xi^{-1}, z) & \text{if } r = 0, i = 0 \pmod N \\ X_{11}(\xi^{i-1}, \xi^{-1}q^r, z) + \frac{e^{i/2Ln\xi}q^{r/2}}{\xi^i - q^r}c & \text{otherwise}\end{cases}$$

for $r \in \mathbb{Z}$ and $0 \le i \le N-1$, then the above two identities can be written as one identity

$$(4.9)\qquad [\bar{X}_{11}(\xi^{i-1}, \xi^{-1}q^r, z_1), \bar{X}_{11}(\xi^{j-1}, \xi^{-1}q^s, z_2)]$$

$$= \bar{X}_{11}(\xi^{i+j-1}, \xi^{-1}q^{r+s}, \xi^{-j}z_1)\delta\Big(\frac{\xi^j z_2}{q^r z_1}\Big) - \bar{X}_{11}(\xi^{i+j-1}, \xi^{-1}q^{r+s}, \xi^{-i}z_2)\delta\Big(\frac{\xi^i z_1}{q^s z_2}\Big)$$
$$+\delta_{r+s,0}\delta_{\overline{i+j},0}(D\delta)\Big(\frac{\xi^j z_2}{q^r z_1}\Big)c.$$

Comparing this with the identity (1.20), we get the following result which was given in [BS] for the $N = 2$ case and in [G2] for arbitrary N.

Corollary 4.10: *Let $M = 1$ and let $G = \langle \xi, q\rangle$ be an admissible subgroup of $\mathbb{C}^\times$ generated by q with $q \neq 0$ not a root of unity, and ξ is an N-th primitive root of unity for $N \geq 2$. Then the Lie algebra $\mathcal{G}(G, M)$ of operators, acting on V_1, gives a representation of the algebra $\widehat{gl_N}(\mathbb{C}_{q^N})$. The representation is given by the mapping*

$$F^iE^m \otimes t_0^m t_1^r \mapsto \begin{cases} x_{11}(m, \xi^{i-1}, \xi^{-1}q^r) + \delta_{m,0}\frac{e^{i/2Ln\xi}q^{r/2}}{\xi^i - q^r}c & \text{if } i \neq 0 (\operatorname{mod} N) \text{ or } r \neq 0, \\ x_{11}(m, \xi^{-1}, \xi^{-1}) & \text{if } i = 0 (\operatorname{mod} N) \text{ and } r = 0, \end{cases}$$
$$c_0 \mapsto \frac{c}{N}, c_1 \mapsto 0$$

for $r \in \mathbb{Z}$ and $0 \leq i \leq N-1$.

Remark 4.11: In general, let $M, N \geq 2$ be integers, $G = \langle \xi, q_1, \ldots, q_\nu\rangle$ an admissible subgroup of $\mathbb{C}^\times$ with finitely many generators, where $q_1, \ldots, q_\nu$ are the free generators of G and ξ is an N-th root of unity. Then the Lie algebra $\mathcal{G}(G, M)$ of operators, acting on the Fock space V_M, gives a representation to the Lie algebra $\widehat{gl_{MN}}(\mathbb{C}_Q)$ where the quantum torus $\mathbb{C}_Q = \mathbb{C}_Q[t_0^{\pm 1}, t_1^{\pm 1}, \ldots, t_\nu^{\pm 1}]$ is determined by the matrix $Q = (q_{ij})_{(\nu+1)\times(\nu+1)}$ with $q_{i0} = q_i^N, q_{0i} = q_i^{-N}$ for $1 \leq i \leq \nu$, and $q_{ij} = 1$ for all other values of i, j.

In particular, if $\nu = 1$, that is $G = \langle \xi, q\rangle$, then the Lie algebra $\mathcal{G}(G, M)$ is generated by the coefficient operators of the vertex operators $X_{ij}(\xi^{k-1}, \xi^{-1}q^r, z)$ for $1 \leq i, j \leq M$, $1 \leq k \leq N-1$ and $r \in \mathbb{Z}$. Moreover, from Theorem 3.3, we have:

(i) if $\overline{k+k'} \neq 0$ or $r+s \neq 0$, then

$$\begin{aligned}
&[X_{ij}(\xi^{k-1},\xi^{-1}q^r,z_1),X_{i'j'}(\xi^{k'-1},\xi^{-1}q^{r'},z_2)]\\
=&\delta_{ji'}X_{ij'}(\xi^{k-1},\xi^{-1-k'}q^{r+r'},z_1)\delta\Big(\frac{\xi^{k'}z_2}{q^r z_1}\Big)\\
&-\delta_{j'i}X_{i'j}(\xi^{k'-1},\xi^{-1-k}q^{r+r'},z_2)\delta\Big(\frac{\xi^k z_1}{q^{r'}z_2}\Big)\\
&+\frac{e^{\frac{k+k'}{2}Ln\xi}q^{\frac{r+r'}{2}}}{\xi^{k+k'}-q^{r+r'}}\delta_{ji'}\delta_{j'i}\Big(\delta\Big(\frac{\xi^{k'}z_2}{q^r z_1}\Big)-\delta\Big(\frac{\xi^k z_1}{q^{r'}z_2}\Big)\Big)c\\
=&\delta_{ji'}X_{ij'}(\xi^{k+k'-1},\xi^{-1}q^{r+r'},\xi^{-k'}z_1)\delta\Big(\frac{\xi^{k'}z_2}{q^r z_1}\Big)\\
&-\delta_{j'i}X_{i'j}(\xi^{k+k'-1},\xi^{-1}q^{r+r'},\xi^{-k}z_2)\delta\Big(\frac{\xi^k z_1}{q^{r'}z_2}\Big)\\
&+\frac{e^{\frac{k+k'}{2}Ln\xi}q^{\frac{r+r'}{2}}}{\xi^{k+k'}-q^{r+r'}}\delta_{ji'}\delta_{j'i}\Big(\delta\Big(\frac{\xi^{k'}z_2}{q^r z_1}\Big)-\delta\Big(\frac{\xi^k z_1}{q^{r'}z_2}\Big)\Big)c,
\end{aligned}$$

and

(ii) if $\overline{k+k'} = 0$ and $r+s=0$, then

$$\begin{aligned}
&[X_{ij}(\xi^{k-1},\xi^{-1}q^r,z_1),X_{i'j'}(\xi^{k'-1},\xi^{-1}q^{r'},z_2)]\\
=&\delta_{ji'}X_{ij'}(\xi^{k-1},\xi^{-1-k'}q^{r+r'},z_1)\delta\Big(\frac{\xi^{k'}z_2}{q^r z_1}\Big)\\
&-\delta_{j'i}X_{i'j}(\xi^{k'-1},\xi^{-1-k}q^{r+r'},z_2)\delta\Big(\frac{\xi^k z_1}{q^{r'}z_2}\Big)+\delta_{ji'}\delta_{j'i}(D\delta)\Big(\frac{\xi^{k'}z_2}{q^r z_1}\Big)c\\
=&\delta_{ji'}X_{ij'}(\xi^{-1},\xi^{-1},\xi^{-k'}z_1)\delta\Big(\frac{\xi^{k'}z_2}{q^r z_1}\Big)\\
&-\delta_{j'i}X_{i'j}(\xi^{-1},\xi^{-1},\xi^{-k}z_2)\delta\Big(\frac{\xi^k z_1}{q^{r'}z_2}\Big)+\delta_{ji'}\delta_{j'i}(D\delta)\Big(\frac{\xi^{k'}z_2}{q^r z_1}\Big)c.
\end{aligned}$$

Set

$$\overline{X}_{ij}(\xi^{k-1},\xi^{-1}q^r,z) = \begin{cases} X_{ij}(\xi^{k-1},\xi^{-1}q^r,z)+\delta_{ij}\frac{e^{k/2Ln\xi}q^{r/2}}{\xi^k-q^r}c & \text{if } k\neq 0(\operatorname{mod} N) \text{ or } r\neq 0,\\ X_{ij}(\xi^{-1},\xi^{-1},z) & \text{if } k=0(\operatorname{mod} N) \text{ and } r=0.\end{cases}$$

Then we have

$$[\overline{X}_{ij}(\xi^{k-1},\xi^{-1}q^r,z_1),\overline{X}_{i'j'}(\xi^{k'-1},\xi^{-1}q^{r'},z_2)]$$

$$
\begin{aligned}
=&\delta_{ji'}\overline{X}_{ij'}(\xi^{k+k'-1},\xi^{-1}q^{r+r'},\xi^{-k'}z_1)\delta\Big(\frac{\xi^{k'}z_2}{q^r z_1}\Big)\\
&-\delta_{j'i}\overline{X}_{i'j}(\xi^{k+k'-1},\xi^{-1}q^{r+r'},\xi^{-k}z_2)\delta\Big(\frac{\xi^{k}z_1}{q^{r'} z_2}\Big)\\
&+\delta_{ji'}\delta_{j'i}\delta_{\overline{k+k'},0}\delta_{r+r',0}(D\delta)\Big(\frac{\xi^{k'}z_2}{q^r z_1}\Big)c.
\end{aligned}
$$

Comparing this identity with identity (1.21), we get

COROLLARY 4.12: *Let $M, N \geq 2$, and let $G = \langle \xi, q\rangle$ be an admissible subgroup of $\mathbb{C}^\times$ generated by q with $q \neq 0$ not a root of unity, and ξ an N-th primitive root of unity. Then the Lie algebra $\mathcal{G}(G, M)$ of operators, acting on V_1, gives a representation of the algebra $\widehat{gl_{MN}}(\mathbb{C}_{q^N})$, and the representation is given by the mapping*

$$
\begin{aligned}
&E_{ij}\otimes F^kE^m\otimes t_0^m t_1^r\\
\mapsto &\begin{cases} x_{ij}(m,\xi^{k-1},\xi^{-1}q^r)+\delta_{m,0}\delta_{ij}\frac{e^{k/2Ln\xi}q^{r/2}}{\xi^i-q^r}c & \text{if } k\neq 0 (\operatorname{mod} N) c \text{ or } r\neq 0,\\ x_{ij}(m,\xi^{-1},\xi^{-1}) & \text{if } k=0(\operatorname{mod} N) \text{ and } r=0,\end{cases}\\
&c_0\mapsto \frac{c}{N}, c_1\mapsto 0
\end{aligned}
$$

for $r \in \mathbb{Z}$ and $0 \leq k \leq N-1$.

The Lie algebra $\mathcal{G}(G, M)$ given in the previous corollary contains two interesting subalgebras which give representations to the Lie algebras $\widehat{gl_N}(\mathbb{C}_{q^N})$ and $\widehat{gl_M}(\mathbb{C}_q)$. Moreover, we will see that these two subalgebras contain subalgebras that give representations to the affine algebras $\widehat{gl_N}(\mathbb{C})$ of level M and $\widehat{gl_M}(\mathbb{C})$ of level N, respectively. Indeed, for $a, b \in G = \langle \xi, q\rangle$, let

$$Y(a,b,z)=\sum_{k=1}^{M} X_{kk}(a,b,z) \tag{4.13}$$

and formally write

$$Y(a,b,z)=\sum_{m\in\mathbb{Z}} y(m,a,b)z^{-m}. \tag{4.14}$$

Let $\mathcal{L}_1$ be the Lie algebra generated by all of the coefficients of $Y(a, b, z)$ for $a, b \in G$. We note that

$$Y(\xi^i q^r,\xi^j q^s,z)=Y(\xi^{i-j-1},\xi^{-1}q^{s-r},\xi^{j+1}q^r z), \tag{4.15}$$

so $\mathcal{L}_1$ is indeed generated by the coefficients of the vertex operators with the form $Y(\xi^{i-1},\xi^{-1}q^r,z)$ for $r\in\mathbb{Z}$ and $0\le i\le N-1$. Moreover, applying Theorem 3.3, we have, if $r+s\neq 0$ or $\overline{i+j}\neq 0(\operatorname{mod} N)$,

$$\begin{aligned}(4.16)\qquad &[Y(\xi^{i-1},\xi^{-1}q^r,z_1),Y(\xi^{j-1},\xi^{-1}q^s,z_2)]\\ =&\sum_{k=1}^{M}[X_{kk}(\xi^{i-1},\xi^{-1}q^r,z_1),X_{kk}(\xi^{j-1},\xi^{-1}q^s,z_2)]\\ =&\sum_{k=1}^{M}\Big(X_{kk}(\xi^{i+j-1},\xi^{-1}q^{r+s},\xi^{-j}z_1)\delta\Big(\frac{\xi^j z_2}{q^r z_1}\Big)\\ &-X_{kk}(\xi^{i+j-1},\xi^{-1}q^{r+s},\xi^{-j}z_2)\delta\Big(\frac{\xi^i z_1}{q^s z_2}\Big)\\ &+\frac{e^{\frac{i+j}{2}Ln\xi}q^{\frac{r+s}{2}}}{\xi^{i+j}-q^{r+s}}\Big(\delta\Big(\frac{\xi^j z_2}{q^r z_1}\Big)-\delta\Big(\frac{\xi^i z_1}{q^s z_2}\Big)\Big)\Big)\\ =&Y(\xi^{i+j-1},\xi^{-1}q^{r+s},\xi^{-j}z_1)\delta\Big(\frac{\xi^j z_2}{q^r z_1}\Big)-Y(\xi^{i+j-1},\xi^{-1}q^{r+s},\xi^{-j}z_2)\delta\Big(\frac{\xi^i z_1}{q^s z_2}\Big)\\ &+\frac{e^{\frac{ij}{2}Ln\xi}q^{\frac{r+s}{2}}}{\xi^{i+j}-q^{r+s}}\Big(\delta\Big(\frac{\xi^j z_2}{q^r z_1}\Big)-\delta\Big(\frac{\xi^i z_1}{q^s z_2}\Big)\Big),\end{aligned}$$

while if $r+s=0$ and $\overline{i+j}=0(\operatorname{mod} N)$, then

$$\begin{aligned}(4.17)\qquad &[Y(\xi^{i-1},\xi^{-1}q^r,z_1),Y(\xi^{j-1},\xi^{-1}q^s,z_2)]\\ &=\sum_{k=1}^{M}[X_{kk}(\xi^{i-1},\xi^{-1}q^r,z_1),X_{kk}(\xi^{j-1},\xi^{-1}q^s,z_2)]\\ =&\sum_{k=1}^{M}\Big(X_{kk}(\xi^{-1},\xi^{-1},\xi^{-j}z_1)\delta\Big(\frac{\xi^j z_2}{q^r z_1}\Big)\\ &-X_{kk}(\xi^{-1},\xi^{-1},\xi^{-j}z_2)\delta\Big(\frac{\xi^i z_1}{q^s z_2}\Big)+(D\delta)\Big(\frac{\xi^j z_2}{q^r z_1}\Big)\Big)\\ =&Y(\xi^{-1},\xi^{-1},\xi^{-j}z_1)\delta\Big(\frac{\xi^j z_2}{q^r z_1}\Big)-Y(\xi^{-1},\xi^{-1},\xi^{-j}z_2)\delta\Big(\frac{\xi^i z_1}{q^s z_2}\Big)+M(D\delta)\Big(\frac{\xi^j z_2}{q^r z_1}\Big).\end{aligned}$$

Therefore, if we define

$$\bar{Y}(\xi^{i-1},\xi^{-1}q^r,z)=\begin{cases}Y(\xi^{i-1},\xi^{-1}q^r,z)+M\frac{e^{i/2Ln\xi}q^{r/2}}{\xi^i-q^r}c & \text{if } r\neq 0 \text{ or } \bar{i}\neq 0,\\ Y(\xi^{-1},\xi^{-1},z) & \text{if } r=0 \text{ and } \bar{i}=0,\end{cases}$$

then we can rewrite the two identities (4.16) and (4.17) as just one identity

$$(4.18)\qquad [\bar{Y}(\xi^{i-1},\xi^{-1}q^r,z_1),\bar{Y}(\xi^{j-1},\xi^{-1}q^s,z_2)]$$

$$= \bar{Y}(\xi^{-1}, \xi^{-1}, \xi^{-j}z_1)\delta\Big(\frac{\xi^j z_2}{q^r z_1}\Big)\bar{Y}(\xi^{-1}, \xi^{-1}, \xi^{-j}z_2)\delta\Big(\frac{\xi^i z_1}{q^s z_2}\Big) + M(D\delta)\Big(\frac{\xi^j z_2}{q^r z_1}\Big)c.$$

Therefore we have the following result:

PROPOSITION 4.19: *The Lie algebra $\mathcal{L}_1$ of operators acting on V_M gives a representation of the Lie algebra $\widehat{gl_N}(\mathbb{C}_{q^N})$, and the representation is given by the mapping*

$$F^iE^m \otimes t_0^m t_1^r \mapsto \begin{cases} y(m, \xi^{i-1}, \xi^{-1}q^r) + M\frac{e^{\frac{i}{2}Ln\xi}q^{r/2}}{\xi^i - q^r}\delta_{m,0}c & \\ \qquad\qquad \text{if } i \neq 0 (\text{mod } N) \text{ or } r \neq 0, & \\ y(m, 1, 1) & \\ \qquad\qquad \text{if } i = 0 (\text{mod } N) \text{ and } r = 0, & \end{cases}$$
$$c_0 \mapsto Mc, c_1 \mapsto 0$$

for $r \in \mathbb{Z}$ and $0 \le i \le N-1$.

Recall from Corollary 4.7 that the Fock space V_M affords a representation of the Lie algebra $\mathcal{G}(\langle q\rangle, M) \subset \mathcal{G}(\langle \xi, q\rangle, M)$, where ξ, q are given in Corollary 4.10, and

$$\mathcal{G}(\langle q\rangle, M) = \text{span}\{c \text{ and } x_{ij}(m, 1, q^r)| \text{ for } m, r \in \mathbb{Z}, 1 \le i, j \le M\}.$$

Now we define a subalgebra of $\mathcal{G}(\langle q\rangle, M) \subset \mathcal{G}(\langle \xi, q\rangle, M)$

$$\mathcal{L}_2 = \text{span}\{c \text{ and } x_{ij}(Nm, 1, q^r)| \text{ for } m, r \in \mathbb{Z}, 1 \le i, j \le M\}.$$

Then we have

PROPOSITION 4.20: *$\mathcal{L}_2$ forms a Lie subalgebra of $\mathcal{G}(\langle q\rangle, M)$, and $\mathcal{L}_2$ is also isomorphic to $\mathcal{G}(\langle q\rangle, M)$ via the isomorphism given by*

$$x_{ij}(m, 1, q^r) \mapsto x_{ij}(Nm, 1, q^r), c \mapsto Nc.$$

Therefore $\mathcal{L}_2$ gives a representation of $\widehat{gl_N}(\mathbb{C}_{q^N})$.

PROPOSITION 4.21: *For $m, n, r, s, \in \mathbb{Z}$ and $i \neq 0$ (mod N), $1 \le k \neq l \le M$, we have*

(4.22)
$$[y(m, \xi^{i-1}, \xi^{-1}q^r), x_{kl}(Nn, 1, q^s)] = (q^{rNn} - q^{sm})x_{kl}(m + Nn, \xi^{i-1}, \xi^{-1}q^{r+s}).$$

Proof: We apply Theorem 3.3 to obtain

$$[Y(\xi^{i-1},\xi^{-1}q^r,z_1),X_{kl}(1,q^s,z_2)]=\Big[\sum_{j=1}^{M}X_{jj}(\xi^{i-1},\xi^{-1}q^r,z_1),X_{kl}(1,q^s,z_2)\Big]$$

$$=\sum_{j=1}^{M}\Big\{X_{jl}(\xi^{i-1},\xi^{-1}q^{r+s},z_1)\delta_{jk}\delta(\frac{z_2}{\xi^{-1}q^rz_1})-X_{kj}(1,\xi^{-i}q^{r+s},z_2)\delta_{jl}\delta\Big(\frac{\xi^{i-1}z_1}{q^sz_2}\Big)\Big\}$$

$$=X_{kl}(\xi^{i-1},\xi^{-1}q^{r+s},z_1)\delta\Big(\frac{z_2}{\xi^{-1}q^rz_1}\Big)-X_{kl}(1,\xi^{-i}q^{r+s},z_2)\delta\Big(\frac{\xi^{i-1}z_1}{q^sz_2}\Big)$$

$$=X_{kl}(\xi^{i-1},\xi^{-1}q^{r+s},z_1)\delta\Big(\frac{z_2}{\xi^{-1}q^rz_1}\Big)-X_{kl}(\xi^{i-1},\xi^{-1}q^{r+s},q^{-s}z_1)\delta\Big(\frac{\xi^{i-1}z_1}{q^sz_2}\Big).$$

This then gives

$$[y(m,\xi^{i-1},\xi^{-1}q^r),x_{kl}(n,1,q^s)]=\xi^{-n}(q^{rn}-q^{sm}\xi^{in})x_{kl}(m+n,\xi^{i-1},\xi^{-1}q^{r+s}),$$

which immediately implies (4.22). ∎

Remark 4.23: Let $\mathcal{G}_i\subset\mathcal{L}_i\subset\mathcal{G}(\langle\xi,q\rangle,M)$, $i=1,2$, be such that

$$\mathcal{G}_1=\text{span}\{\text{cand}y(m,\xi^{i-1},\xi^{-1})|\text{ for }m\in\mathbb{Z},0\le i\le N-1\},$$
$$\mathcal{G}_2=\text{span}\{\text{cand}x_{ij}(Nm,1,1)|\text{ for }m\in\mathbb{Z},1\le i,j\le M\}.$$

Then the two subalgebras $\mathcal{G}_1,\mathcal{G}_2$ of $\mathcal{G}(\langle\xi,q\rangle,M)$ respectively give representations of the affine algebra $\widehat{gl_N}(\mathbb{C})$ of level M and $\widehat{gl_M}(\mathbb{C})$ of level N. Let $\mathcal{G}_i'$ be the derived algebras of $\mathcal{G}_i$. Then we have the so-called dual pair property given in [F1]: $[\mathcal{G}_1',\mathcal{G}_2']=(0)$. However, clearly, we have $[\mathcal{L}_1',\mathcal{L}_2']\neq(0)$.

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